Chapter 6- Variables Control Charts Part I X-bar and R Chart

Learning Objectives

1. Understand the statistical basis of Shewhart control charts for variables
2. Know how to design variables control charts
3. Know how to set up and use $\bar{x}$ and R control charts
4. Know how to estimate process capability from the control chart information
5. Know how to interpret patterns on $\bar{x}$ and R control charts
6. Know how to set up and use $\bar{x}$ and s or s^2 control charts
7. Know how to set up and use control charts for individual measurements
8. Understand the importance of the normality assumption for individuals control charts and know how to check this assumption
9. Understand the rational subgroup concept for variables control charts
10. Determine the average run length for variables control charts

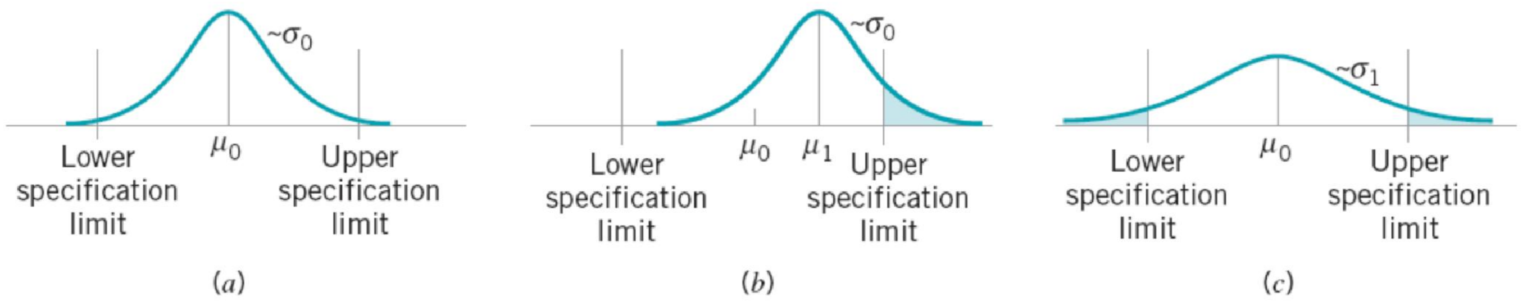
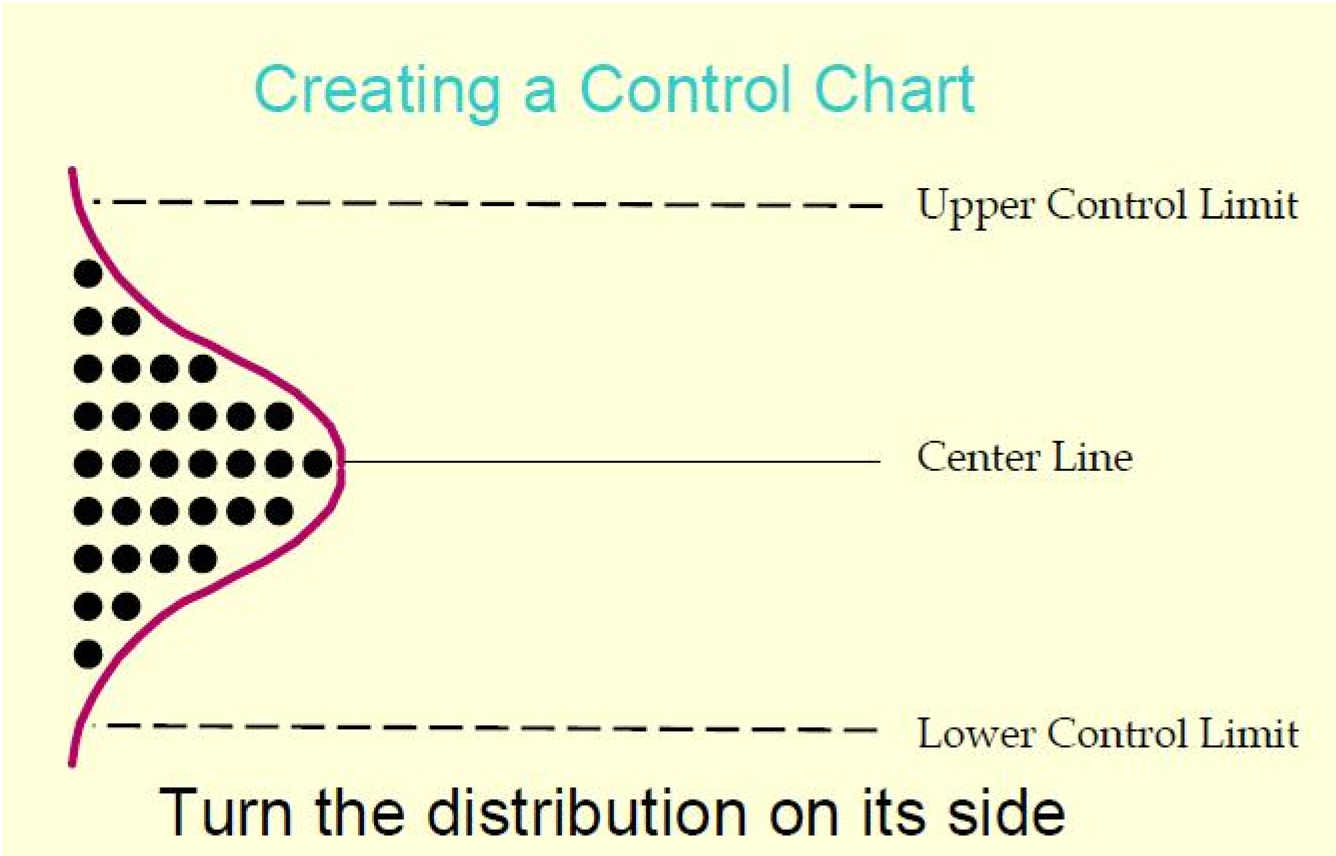
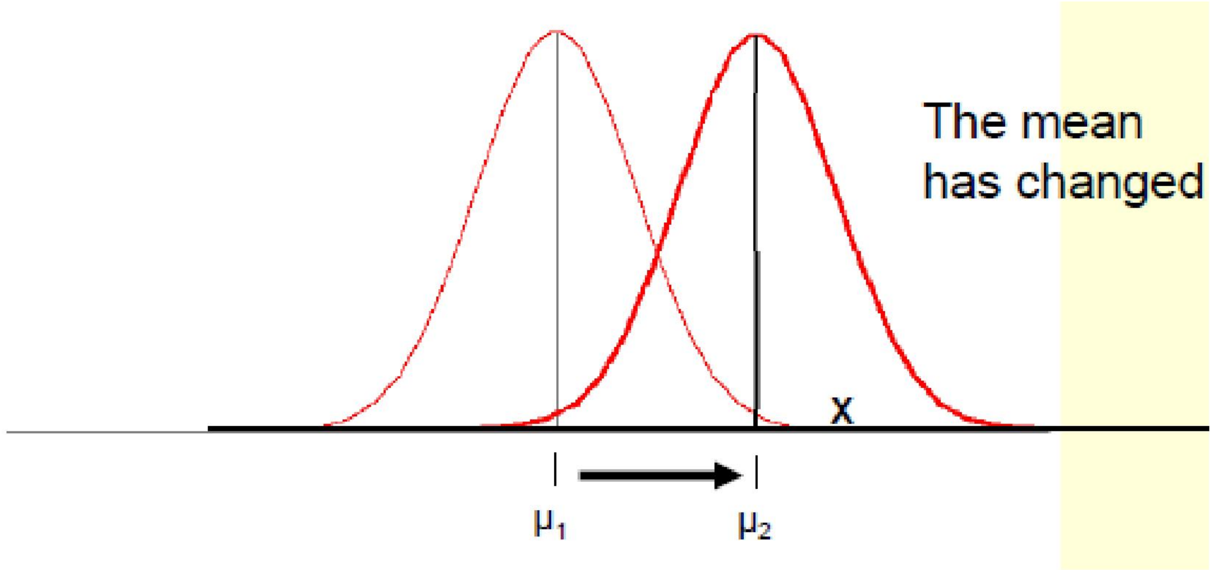


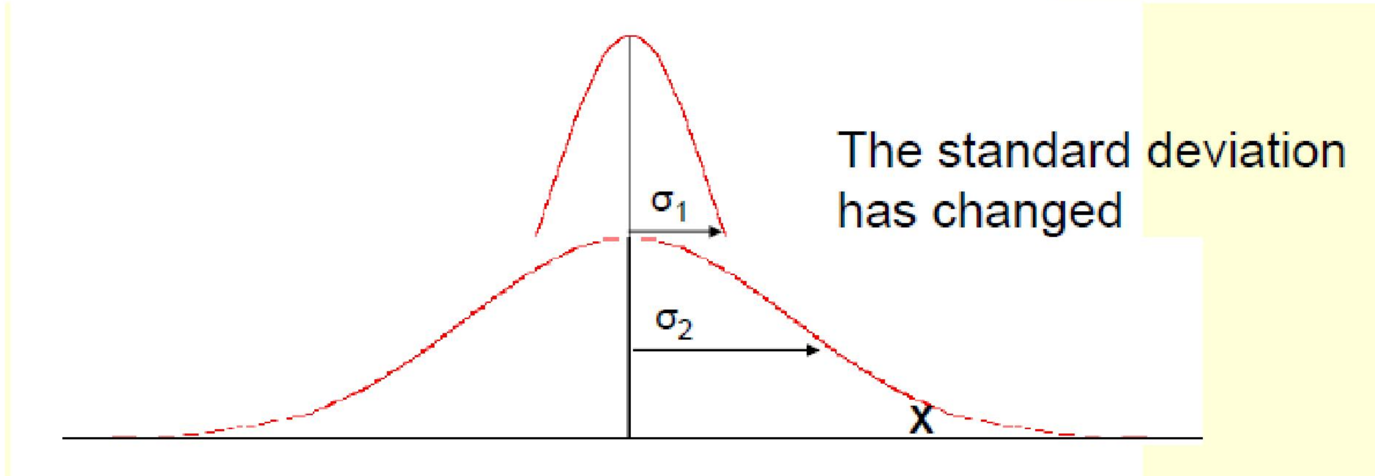
FIGURE 6.1 The need for controlling both process mean and process variability. (*a*) Mean and standard deviation at nominal levels. (*b*) Process mean $\mu_1 > \mu_0$. (*c*) Process standard deviation $\sigma_1 > \sigma_0$.

Role of a Normal Distribution in Variables Control Charts

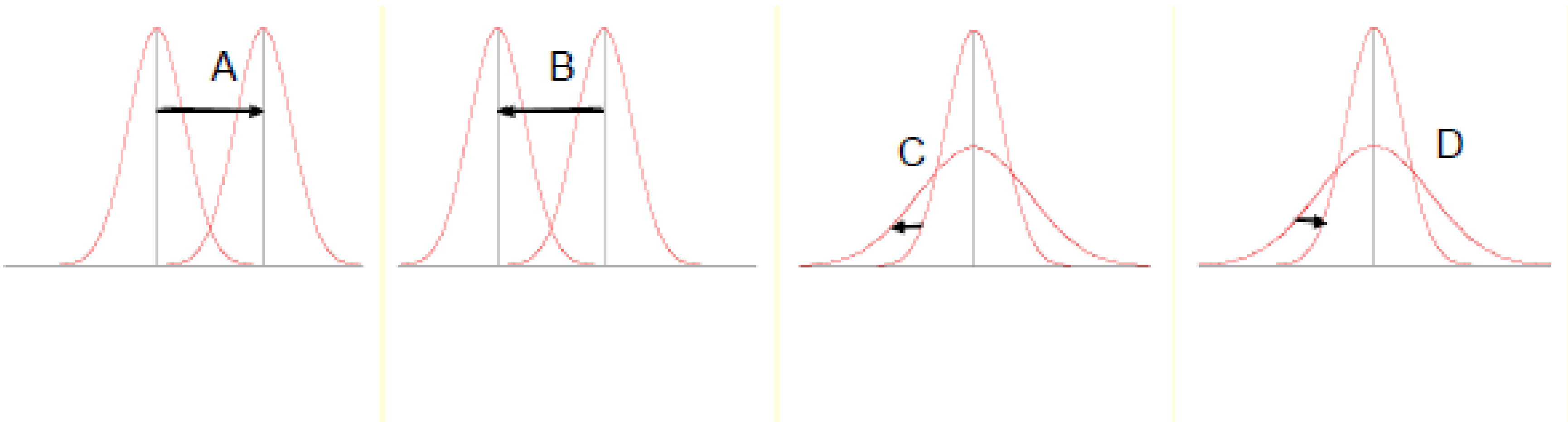


Going out of control

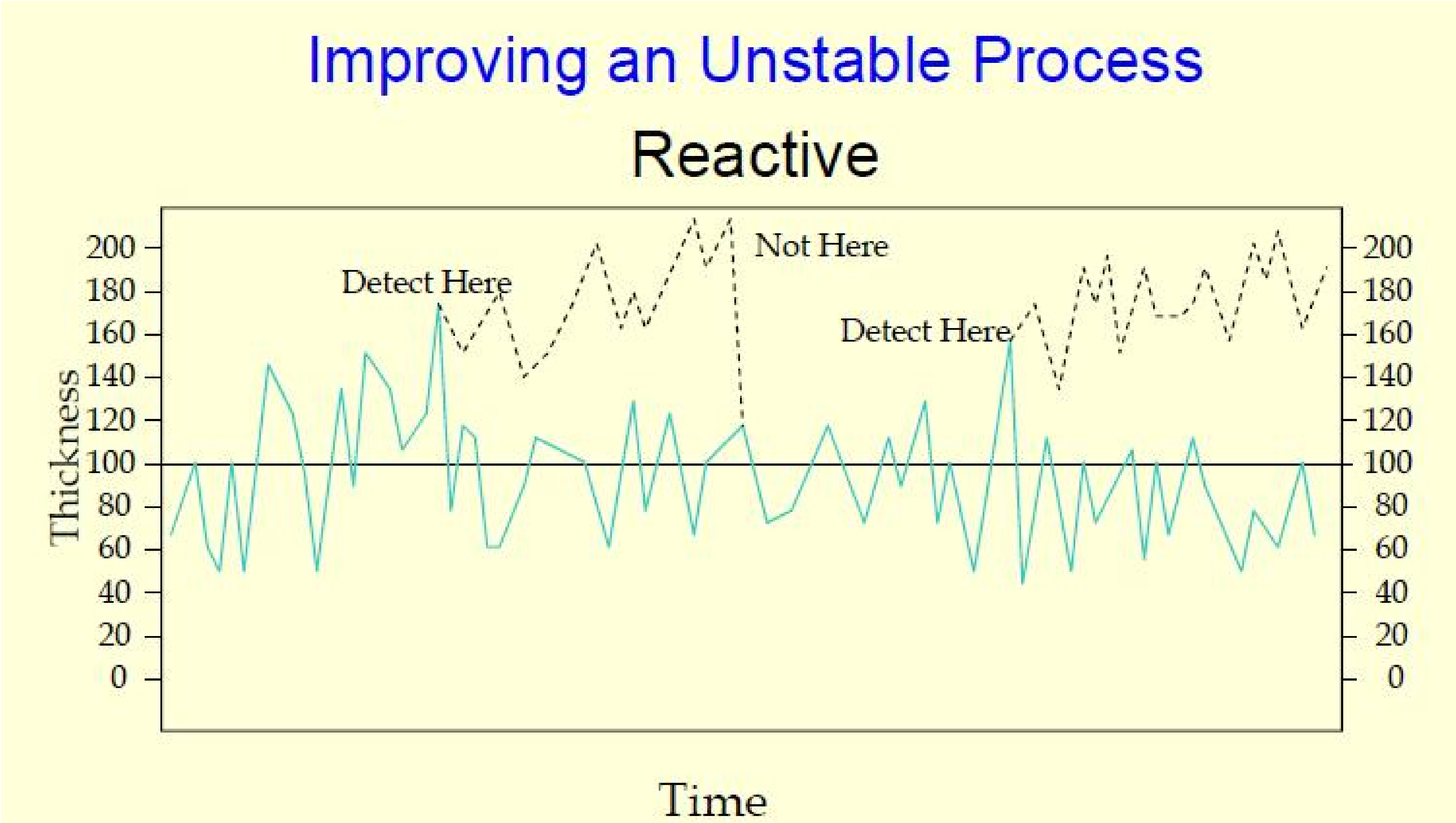




Real time of Charts



Why SPC?



Improving an Unstable Process
Reactive
Detect Here
Not Here
Detect Here
Thickness
200
180
160
140
120
100
80
60
40
20
0
Time

Summary

Control charts for variables

- X-bar and R charts*
- X-bar and S charts
- X-bar and S^2 charts
- X-bar and moving range (MR) charts
- Individual (I) chart
- Moving average (MA) chart
- Exponentially-weighted moving average (EWMA) chart
- Cumulative-sum (CUSUM) chart

Control charts for attributes

- P chart
- NP chart
- C chart
- U chart

6.2 Control Charts for $\bar{x}$ and R

6.2.1 Statistical Basis of the Charts

Suppose that a quality characteristic is normally distributed with mean μ and standard deviation σ, where both μ and σ are known. If $x_1, x_2, \ldots, x_n$ is a sample of size n, then the average of this sample is

$$\boxed{\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}}$$

and we know that $\bar{x}$ is normally distributed with mean μ and standard deviation $\sigma_{\bar{x}} = \sigma/\sqrt{n}$. Furthermore, the probability is $1 - \alpha$ that any sample mean will fall between

$$\boxed{\mu + Z_{\alpha/2}\sigma_{\bar{x}} = \mu + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \mu - Z_{\alpha/2}\sigma_{\bar{x}} = \mu - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}} \tag{6.1}$$

Therefore, if μ and σ are known, equation (6.1) could be used as upper and lower control limits on a control chart for sample means. As noted in Chapter 5, it is customary to replace $Z_{\alpha/2}$ by 3, so that three-sigma limits are employed. If a sample mean falls outside of these limits, it is an indication that the process mean is no longer equal to μ.

Subgroup Data with Unknown μ and σ

In practice, we usually will not know μ and σ. Therefore, they must be estimated from preliminary samples or subgroups taken when the process is thought to be in control. These estimates should usually be based on at least 20 to 25 samples. Suppose that m samples are available, each containing n observations on the quality characteristic. Typically, n will be small, often either 4, 5, or 6. These small sample sizes usually result from the construction of rational subgroups and from the fact that the sampling and inspection costs associated with variables measurements are usually relatively large. Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ be the average of each sample. Then the best estimator of μ, the process average, is the grand average—say,

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_m}{m} \tag{6.2}$$

Thus, $\bar{\bar{x}}$ would be used as the center line on the $\bar{x}$ chart.

To construct the control limits, we need an estimate of the standard deviation σ. Recall from Chapter 4 (Section 4.2) that we may estimate σ from either the standard deviations or the ranges of the m samples. For the present, we will use the range method. If $x_1, x_2, \ldots, x_n$ is a sample of size n, then the range of the sample is the difference between the largest and smallest observations; that is,

$$R = x_{\max} - x_{\min}$$

Let $R_1, R_2, \ldots, R_m$ be the ranges of the m samples. The average range is

$$\bar{R} = \frac{R_1 + R_2 + \cdots + R_m}{m} \tag{6.3}$$

Control Limits for the $\bar{x}$ Chart

$$\text{UCL} = \bar{\bar{x}} + A_2\bar{R}$$
$$\text{Center line} = \bar{\bar{x}} \tag{6.4}$$
$$\text{LCL} = \bar{\bar{x}} - A_2\bar{R}$$

The constant A_2 is tabulated for various sample sizes in Appendix Table VI.

Control Limits for the R Chart

$$\text{UCL} = D_4\bar{R}$$
$$\text{Center line} = \bar{R} \tag{6.5}$$
$$\text{LCL} = D_3\bar{R}$$

The constants D_3 and D_4 are tabulated for various values of n in Appendix Table VI.

$$\hat{\sigma} = \frac{\overline{R}}{d_2} \qquad (6.6)$$

If we use $\overline{\overline{x}}$ as an estimator of μ and $\overline{R}/d_2$ as an estimator of σ, then the parameters of the $\overline{x}$ chart are

$$\boxed{\begin{aligned} \text{UCL} &= \overline{\overline{x}} + \frac{3}{d_2\sqrt{n}}\overline{R} \\ \text{Center line} &= \overline{\overline{x}} \\ \text{LCL} &= \overline{\overline{x}} - \frac{3}{d_2\sqrt{n}}\overline{R} \end{aligned}} \qquad (6.7)$$

If we define

$$\boxed{A_2 = \frac{3}{d_2\sqrt{n}}} \qquad (6.8)$$

then equation (6.7) reduces to equation (6.4).

Now consider the R chart. The center line will be $\bar{R}$. To determine the control limits, we need an estimate of σ_R. Assuming that the quality characteristic is normally distributed, $\hat{\sigma}_R$ can be found from the distribution of the $\boxed{\text{relative range } W = R/\sigma.}$ The standard deviation of W, say d_3, is a known function of n. Thus, since

$$R = W\sigma$$

the standard deviation of R is

$$\sigma_R = d_3\sigma$$

Since σ is unknown, we may estimate σ_R by

$$\boxed{\hat{\sigma}_R = d_3 \frac{\bar{R}}{d_2}} \tag{6.9}$$

W = Relative range

d_2 = Mean of W

d_3 = SD of W

Consequently, the parameters of the R chart with the usual three-sigma control limits are

$$\boxed{\begin{aligned} \text{UCL} &= \bar{R} + 3\hat{\sigma}_R = \bar{R} + 3d_3\frac{\bar{R}}{d_2} \\ \text{Center line} &= \bar{R} \\ \text{LCL} &= \bar{R} - 3\hat{\sigma}_R = \bar{R} - 3d_3\frac{\bar{R}}{d_2} \end{aligned}} \tag{6.10}$$

If we let

$$\boxed{D_3 = 1 - 3\frac{d_3}{d_2} \quad \text{and} \quad D_4 = 1 + 3\frac{d_3}{d_2}}$$

equation (6.10) reduces to equation (6.5).

Phase I Application of $\overline{x}$ and *R* Charts

Equation 6.4 and 6.5 are **trial control limits**

- Determined from *m* initial samples
  - Typically 20-25 subgroups of size *n* between 3 and 5
- Out-of-control points should be examined for assignable causes
  - If assignable causes are found, discard points from calculations and revise the trial control limits
  - Continue examination until all points plot in control
  - Adopt resulting trial control limits for use
- If no assignable cause is found, there are two options
  1. Eliminate points as if an assignable cause were found and revise limits
  2. Retain point and consider limits appropriate for control
- If there are many out-of-control points they should be examined for **patterns** that may identify underlying process problems

Example 6.1 The Hard Bake Process

■ TABLE 6.1

Flow Width Measurements (microns) for the Hard-Bake Process

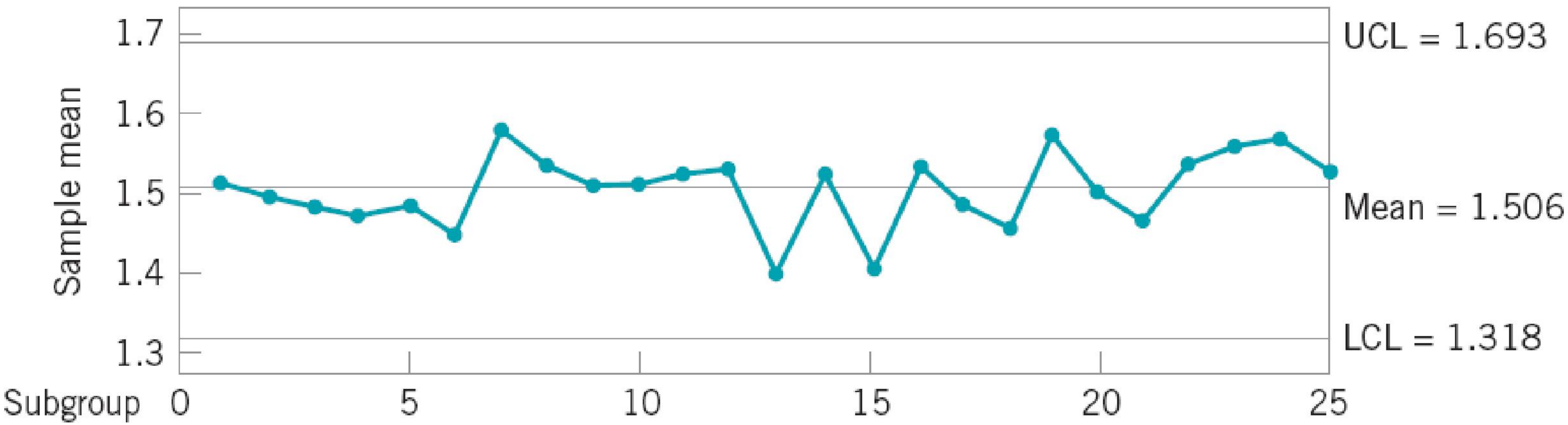
Sample Number	Wafers 1	2	3	4	5	$\bar{x}_i$	R_i
1	1.3235	1.4128	1.6744	1.4573	1.6914	1.5119	0.3679
2	1.4314	1.3592	1.6075	1.4666	1.6109	1.4951	0.2517
3	1.4284	1.4871	1.4932	1.4324	1.5674	1.4817	0.1390
4	1.5028	1.6352	1.3841	1.2831	1.5507	1.4712	0.3521
5	1.5604	1.2735	1.5265	1.4363	1.6441	1.4882	0.3706
6	1.5955	1.5451	1.3574	1.3281	1.4198	1.4492	0.2674
7	1.6274	1.5064	1.8366	1.4177	1.5144	1.5805	0.4189
8	1.4190	1.4303	1.6637	1.6067	1.5519	1.5343	0.2447
9	1.3884	1.7277	1.5355	1.5176	1.3688	1.5076	0.3589
10	1.4039	1.6697	1.5089	1.4627	1.5220	1.5134	0.2658
11	1.4158	1.7667	1.4278	1.5928	1.4181	1.5242	0.3509
12	1.5821	1.3355	1.5777	1.3908	1.7559	1.5284	0.4204
13	1.2856	1.4106	1.4447	1.6398	1.1928	1.3947	0.4470
14	1.4951	1.4036	1.5893	1.6458	1.4969	1.5261	0.2422
15	1.3589	1.2863	1.5996	1.2497	1.5471	1.4083	0.3499
16	1.5747	1.5301	1.5171	1.1839	1.8662	1.5344	0.6823
17	1.3680	1.7269	1.3957	1.5014	1.4449	1.4874	0.3589
18	1.4163	1.3864	1.3057	1.6210	1.5573	1.4573	0.3153
19	1.5796	1.4185	1.6541	1.5116	1.7247	1.5777	0.3062
20	1.7106	1.4412	1.2361	1.3820	1.7601	1.5060	0.5240
21	1.4371	1.5051	1.3485	1.5670	1.4880	1.4691	0.2185
22	1.4738	1.5936	1.6583	1.4973	1.4720	1.5390	0.1863
23	1.5917	1.4333	1.5551	1.5295	1.6866	1.5592	0.2533
24	1.6399	1.5243	1.5705	1.5563	1.5530	1.5688	0.1156
25	1.5797	1.3663	1.6240	1.3732	1.6887	1.5264	0.3224
						$\Sigma\bar{x}_i = 37.6400$	$\Sigma R_i = 8.1302$
						$\bar{\bar{x}} = 1.5056$	$\bar{R} = 0.32521$

$$\bar{\bar{x}} = \frac{\sum_{i=1}^{25} \bar{x}_i}{25} = \frac{37.6400}{25} = 1.5056$$

$$\text{UCL} = \bar{\bar{x}} + A_2\bar{R} = 1.5056 + (0.577)(0.32521) = 1.69325$$

and

$$\text{LCL} = \bar{\bar{x}} - A_2\bar{R} = 1.5056 - (0.577)(0.32521) = 1.31795$$

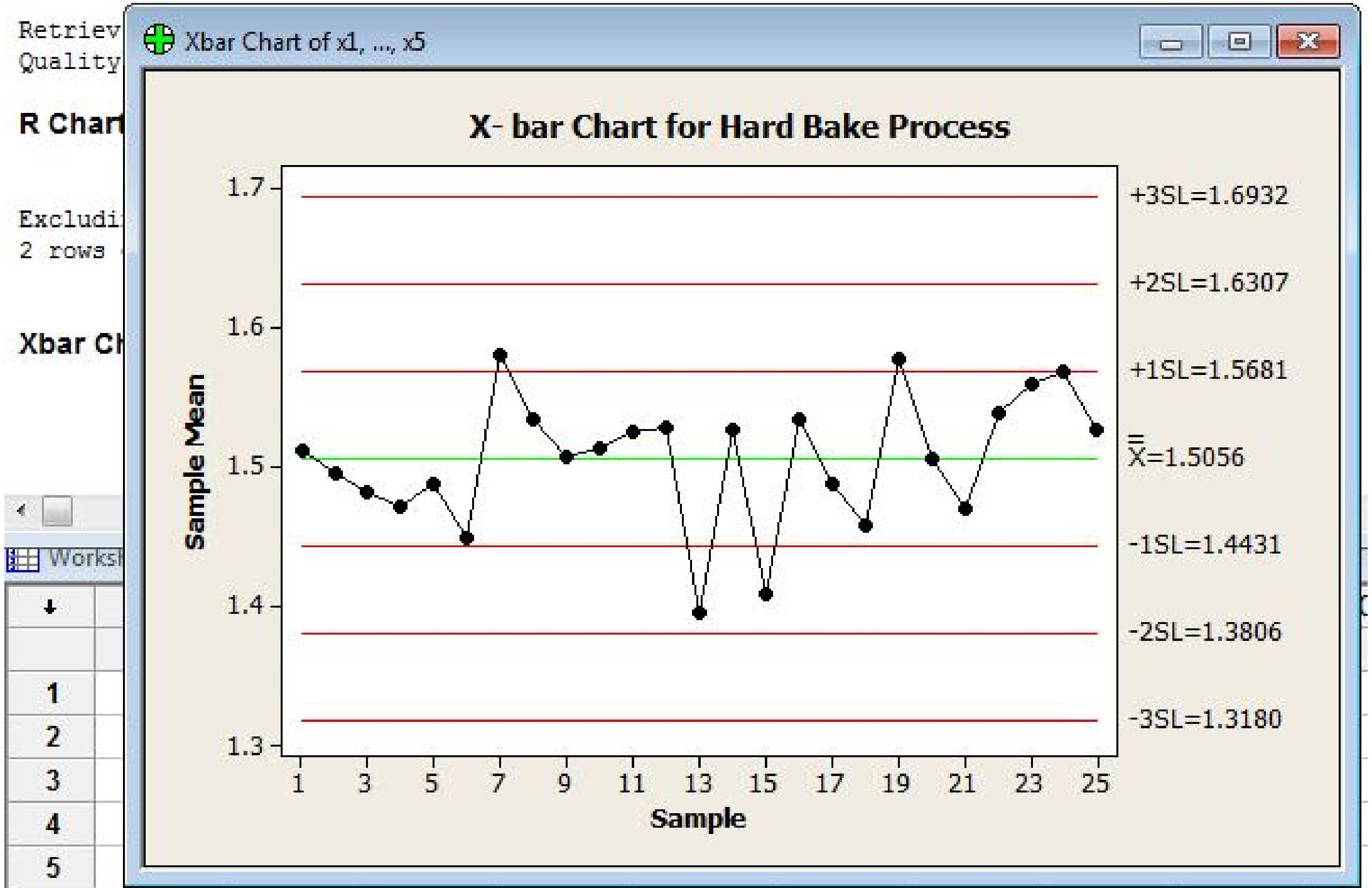


(*a*)

Minitab for X-bar chart

1. Data
2. Choose Stat > Control Charts > Variables Charts for subgroups > xbar.
3. Choose Observations for a subgroup are in one row of columns, then click x1, x2, x3, x4, x5 in the box.
4. Choose Labels, then choose Title and write X-bar chart for Hard Bake Process. Click OK.
5. Click Data Options, then choose Specify which rows to exclude.
6. Click Xbar Options, then click the Estimate tab. Choose Rbar.
7. Click S Limits, then enter 1 2 3 (Not 1, 2, 3).
8. Click the Tests tab. Choose Perform all tests for special causes. Click OK in each dialog box.

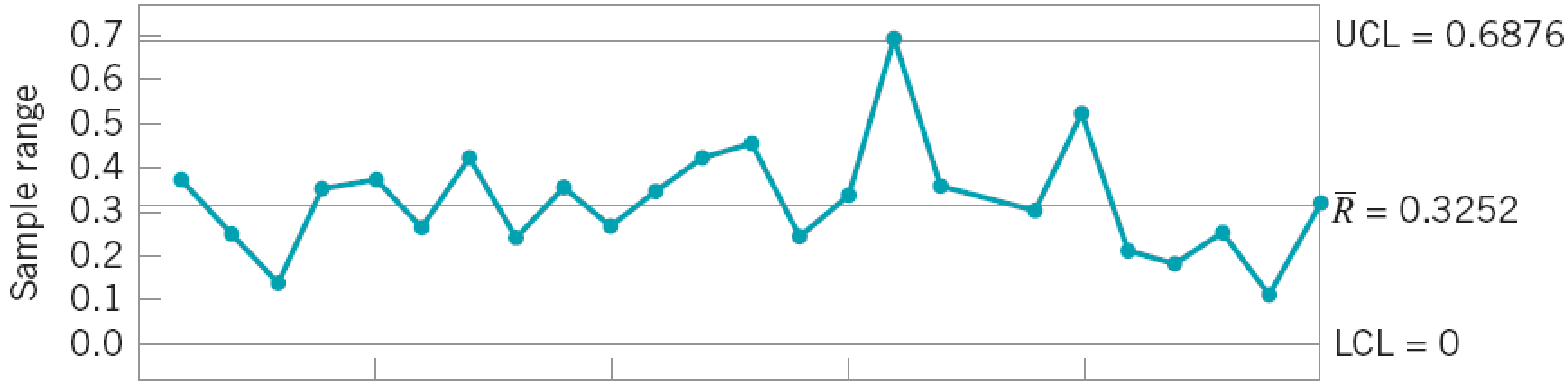
X-bar Chart by Minitab



6	1.5955	1.5451	1.3574	1.3281	1.4198
7	1.6274	1.5064	1.8366	1.4177	1.5144
8	1.4190	1.4303	1.6637	1.6067	1.5519
9	1.3884	1.7277	1.5355	1.5176	1.3688

$$\overline{R} = \frac{\sum_{i=1}^{25} R_i}{25} = \frac{8.1302}{25} = 0.32521$$

$$\text{LCL} = \overline{R}D_3 = 0.32521(0) = 0$$
$$\text{UCL} = \overline{R}D_4 = 0.32521(2.114) = 0.68749$$



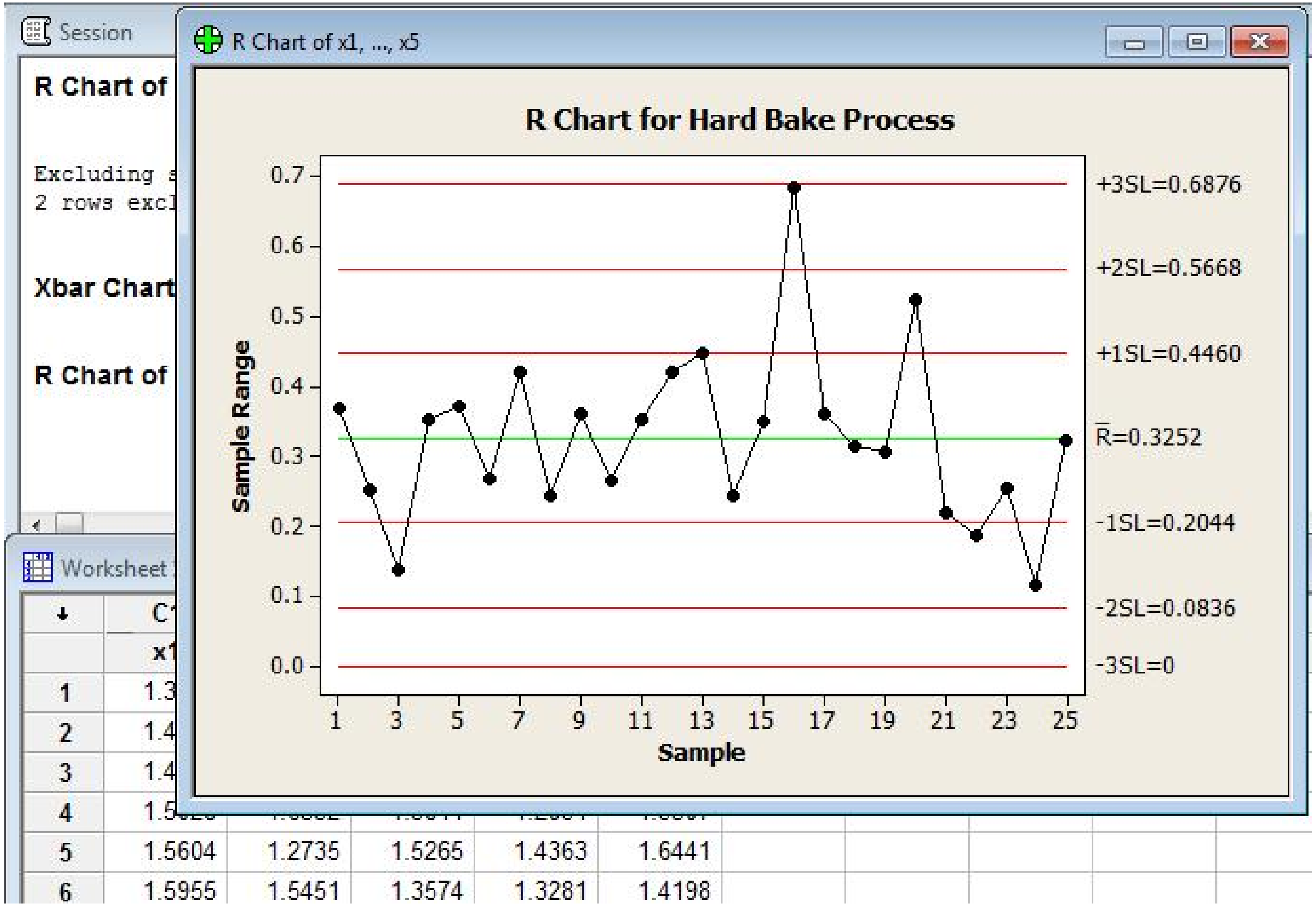
(*b*)

■ **FIGURE 6.2** $\overline{x}$ and R charts (from Minitab) for flow width in the hard-bake process.

Minitab for R Chart

1. Data
2. Choose Stat > Control Charts > Variables Charts for subgroups > R.
3. Choose Observations for a subgroup are in one row of columns, then click x1, x2, x3, x4, x5 in the box.
4. Choose Labels, then choose Title and write R chart for Hard Bake Process. Click OK.
5. Click Data Options, then choose Specify which rows to exclude.
6. Click R Options, then click the Estimate tab. Choose Rbar.
7. Click S Limits, then enter 1 2 3 (Not 1, 2, 3).
8. Click the Tests tab. Choose Perform all tests for special causes. Click OK in each dialog box.

R Chart by Minitab



Estimating Process Capability

The $\bar{x}$ and R charts provide information about the performance or **capability** of the process. From the $\bar{x}$ chart, we may estimate the mean flow width of the resist in the hard-bake process as $\bar{\bar{x}} = 1.5056$ microns. The process standard deviation may be estimated using equation 5-6; that is,

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{0.32521}{2.326} = 0.1398 \text{ microns}$$

where the value of d_2 for samples of size five is found in Appendix Table VI. The specification limits on flow width are 1.50 ± 0.50 microns. The control chart data may be used to describe the capability of the process to produce wafers relative to these specifications. Assuming that flow width is a normally distributed random variable, with mean 1.5056 and standard deviation 0.1398, we may estimate the fraction of nonconforming wafers produced as

$$\begin{aligned} p &= P\{x < 1.00\} + P\{x > 2.00\} \\ &= \Phi\left(\frac{1.00 - 1.5056}{0.1398}\right) + 1 - \Phi\left(\frac{2.00 - 1.5056}{0.1398}\right) \\ &= \Phi(-3.61660) + 1 - \Phi(3.53648) \\ &\simeq 0.00015 + 1 - 0.99980 \\ &\simeq 0.00035 \end{aligned}$$

That is, about 0.035 percent [350 parts per million (ppm)] of the wafers produced will be outside of the specifications.

Another way to express process capability is in terms of the **process capability ratio (PCR) C_p,** which for a quality characteristic with both upper and lower specification limits (USL and LSL, respectively) is

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} \tag{6.11}$$

Note that the 6σ spread of the process is the basic definition of process capability. Since σ is usually unknown, we must replace it with an estimate. We frequently use $\hat{\sigma} = \overline{R}/d_2$ as an estimate of σ, resulting in an estimate $\hat{C}_p$ of C_p. For the hard-bake process, since $\overline{R}/d_2 = \hat{\sigma} = 0.1398$, we find that

$$\hat{C}_p = \frac{2.00 - 1.00}{6(0.1398)} = \frac{1.00}{0.8388} = 1.192$$

This implies that the "natural" tolerance limits in the process (three-sigma above and below the mean) are inside the lower and upper specification limits. Consequently, a moderately small number of nonconforming wafers will be produced. The PCR C_p may be interpreted another way. The quantity

$$P = \left(\frac{1}{C_p}\right)100\%$$

is simply the percentage of the specification band that the process uses up. For the hard-bake process an estimate of P is

$$\hat{P} = \left(\frac{1}{\hat{C}_p}\right)100\% = \left(\frac{1}{1.192}\right)100\% = 83.89$$

That is, the process uses up about 84% of the specification band.

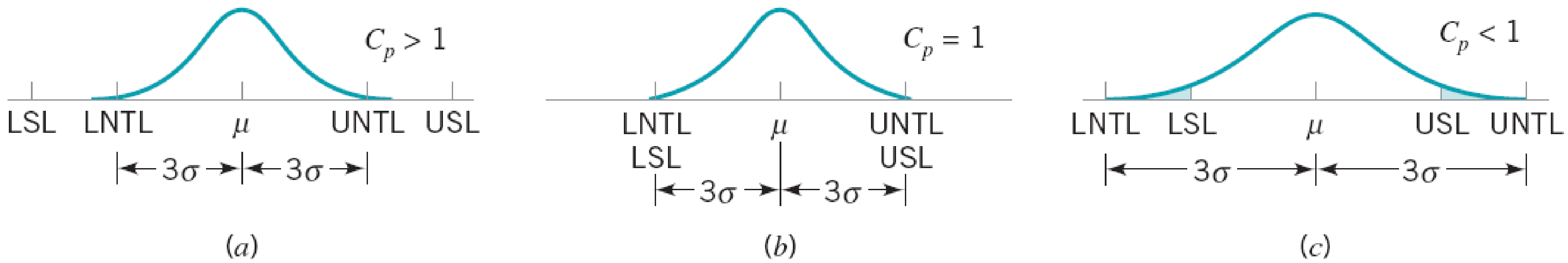
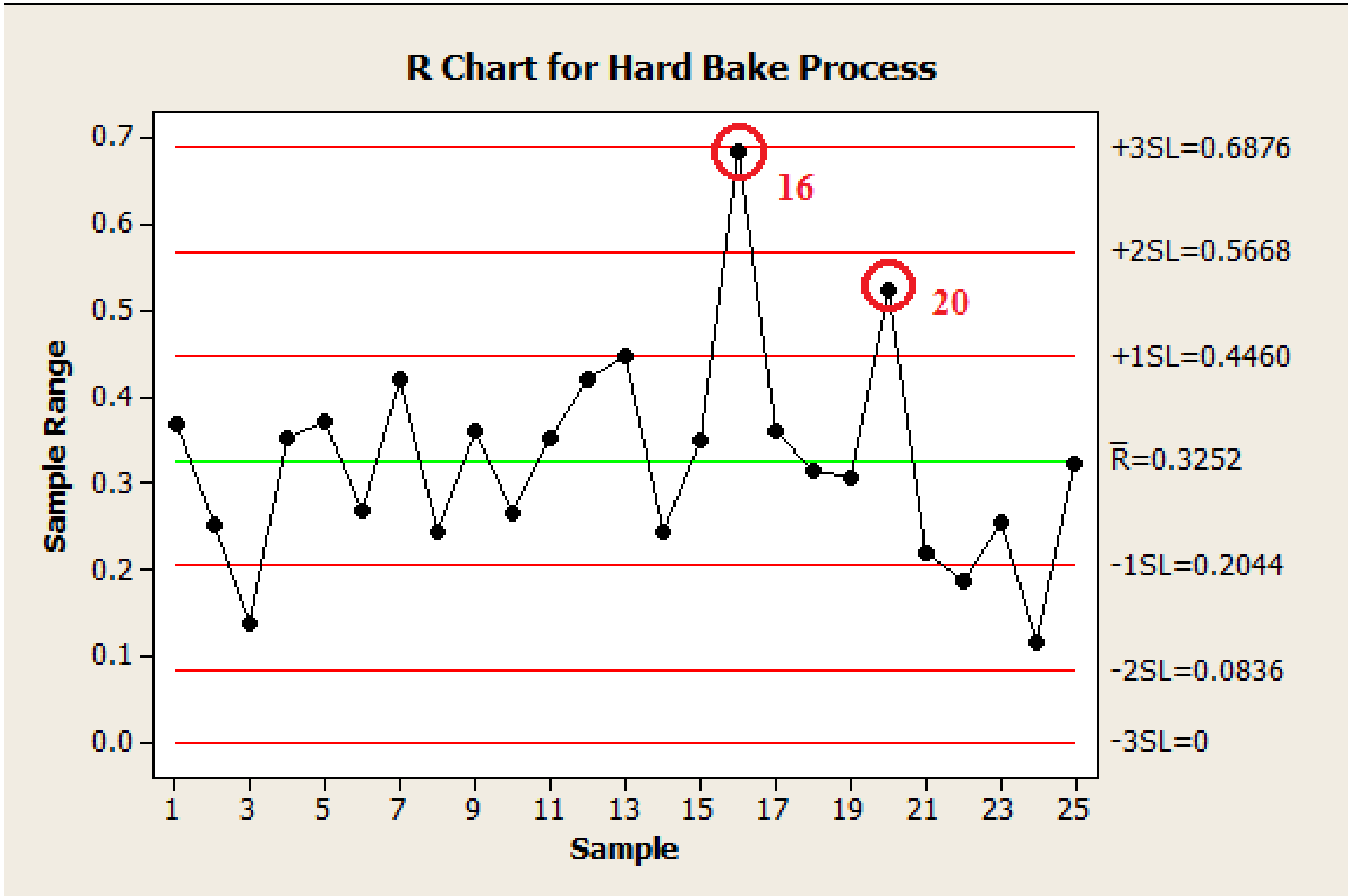


FIGURE 6.3 Process fallout and the process capability ratio C_p.

Revision of Control Limits and Center Lines

- Effective use of control charts requires periodic review and revision of control limits and center lines
- Sometimes users replace the center line on the $\overline{x}$ chart with a target value
- When R chart is out of control, out-of-control points are often eliminated to re-compute a revised value of $\overline{R}$ which is used to determine new limits and center line on R chart and new limits on $\overline{x}$ chart

Revision of Control Limits and Center Lines



Revision of Control Limits

$$\overline{\overline{X}}' = X_0' = \frac{\sum \overline{X} - \sum X_d}{m - m_d}$$

$\sum \overline{X}$ = ผลรวมค่าเฉลี่ยจากรุ่นข้อมูลเดิม

$\sum X_d$ = ผลรวมของค่าเฉลี่ยของข้อมูลที่ตกออกนอกเขตควบคุม และสามารถแก้ไขได้แล้ว (ตัวที่ หาสาเหตุไม่ได้ไม่ต้องเอามาคำควณด้วย)

m = จำนวนกลุ่มข้อมูลเดิม

m_d = จำนวนกลุ่มข้อมูลที่ตกออกนอกเขตพิกัดควบคุม (UCL/LCL)

Revision of Control Limits

$$\bar{R}' = R_0 = \frac{\sum R - \sum R_d}{m - m_d}$$

$$\sigma_0 = \frac{R_0}{d_2}$$

$\overline{X} - Chart$ (Revised)

$$CL_{\overline{X}} = X_0'$$

$$UCL_{\overline{X}} = X_0' + A\sigma_0$$

$$LCL_{\overline{X}} = X_0' - A\sigma_0$$

$$LCL_R = \max.\{0, D_3\left[\frac{d_{2(new)}}{d_{2(Old)}}\right]\overline{R}_{Old}\}$$

$R - Chart$ (Revised)

$$CL_R = R_0$$

$$UCL_R = D_2\sigma_0$$

$$LCL_R = D_1\sigma_0$$

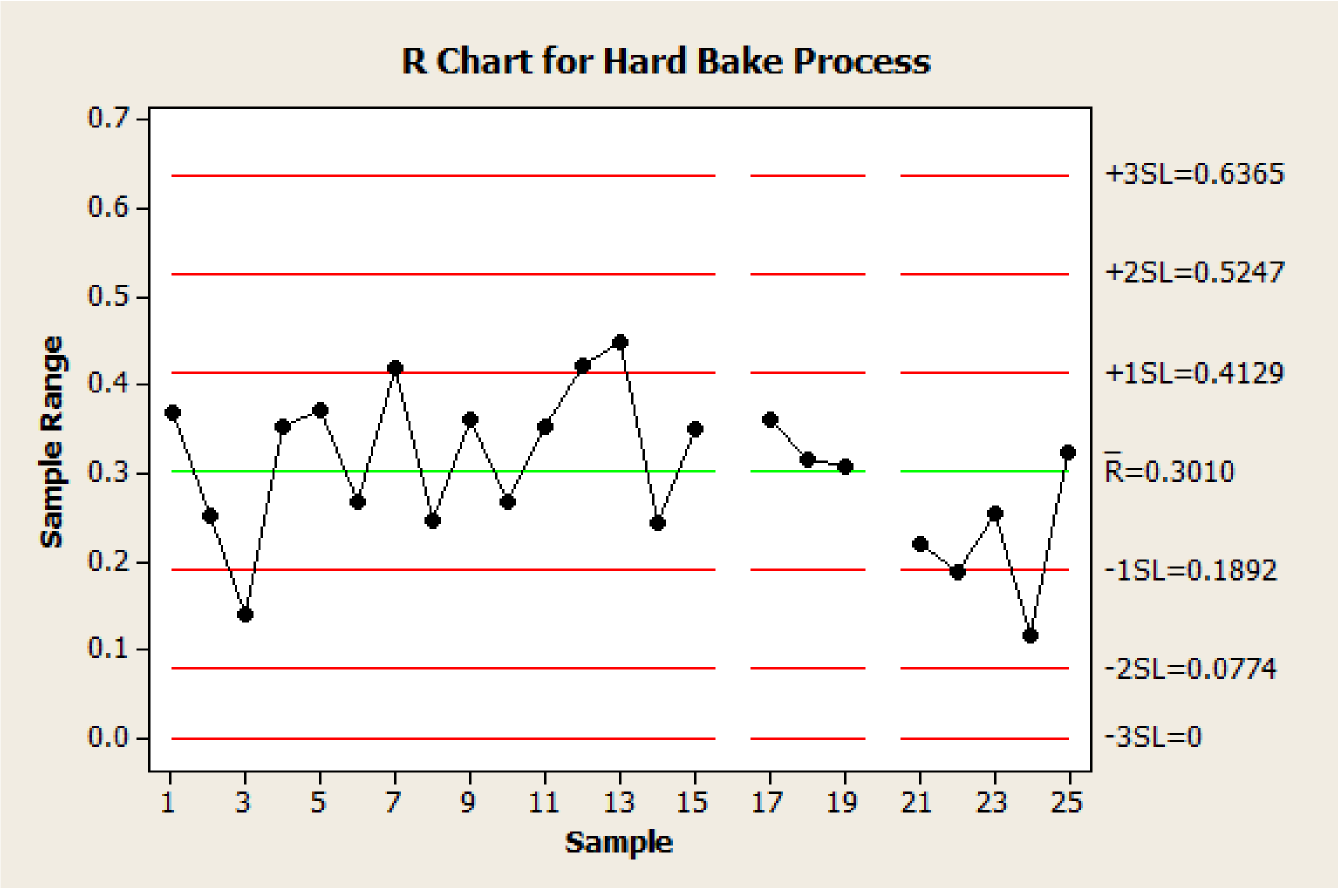
Minitab for Revision Control Chart

Sample Number	x1	x2	x3	x4	x5	x-bar	R	
1	1.3235	1.4128	1.6744	1.4573	1.6914	1.5119	0.3679	
2	1.4314	1.3592	1.6075	1.4666	1.6109	1.4951	0.2517	
3	1.4284	1.4871	1.4932	1.4324	1.5674	1.4817	0.1390	
4	1.5028	1.6352	1.3841	1.2831	1.5507	1.4712	0.3521	
5	1.5604	1.2735	1.5265	1.4363	1.6441	1.4882	0.3706	
6	1.5955	1.5451	1.3574	1.3281	1.4198	1.4492	0.2674	
7	1.6274	1.5064	1.8366	1.4177	1.5144	1.5805	0.4189	
8	1.4190	1.4303	1.6637	1.6067	1.5519	1.5343	0.2447	
9	1.3884	1.7277	1.5355	1.5176	1.3688	1.5076	0.3589	
10	1.4039	1.6697	1.5089	1.4627	1.5220	1.5134	0.2658	
11	1.4158	1.7667	1.4278	1.5928	1.4181	1.5242	0.3509	
12	1.5821	1.3355	1.5777	1.3908	1.7559	1.5284	0.4204	
13	1.2856	1.4106	1.4447	1.6398	1.1928	1.3947	0.4470	
14	1.4951	1.4036	1.5893	1.6458	1.4969	1.5261	0.2422	
15	1.3589	1.2863	1.5996	1.2497	1.5471	1.4083	0.3499	
16	1.5747	1.5301	1.5171	1.1839	1.8662	1.5344	0.6823	Delete
17	1.3680	1.7269	1.3957	1.5014	1.4449	1.4874	0.3589	
18	1.4163	1.3864	1.3057	1.6210	1.5573	1.4573	0.3153	
19	1.5796	1.4185	1.6541	1.5116	1.7247	1.5777	0.3062	
20	1.7106	1.4412	1.2361	1.3820	1.7601	1.5060	0.5240	Delete
21	1.4371	1.5051	1.3485	1.5670	1.4880	1.4691	0.2185	
22	1.4738	1.5936	1.6583	1.4973	1.4720	1.5390	0.1863	
23	1.5917	1.4333	1.5551	1.5295	1.6866	1.5592	0.2533	
24	1.6399	1.5243	1.5705	1.5563	1.5530	1.5688	0.1156	
25	1.5797	1.3663	1.6240	1.3732	1.6887	1.5264	0.3224	
					Sum	37.6403	8.1302	
					Average	1.5056	0.3252	

Minitab for Revision R Chart

1. Data
2. Choose Stat > Control Charts > Variables Charts for Subgroups > R
3. Click Data Options, then choose Specify which rows to exclude.
4. Choose Row numbers, and enter 16 20. Choose Leave Gaps for excluded points.
5. Click OK and OK

Minitab for Revision R Chart



Minitab for Revision X-bar Chart

1. Data
2. Choose Stat > Control Charts > Variables Charts for Subgroups > Xbar
3. Click Data Options, then choose Specify which rows to exclude.
4. Choose Row numbers, and enter 16 20. Choose Leave Gaps for excluded points.
5. Click OK and OK

Minitab for Revision X-bar Chart