Material selection with shape
Common modes of loading and the section-shapes that are chosen to support them: (a) axial tension (b) bending (c) torsion and (d) axial compression, which can lead to buckling.
Selection of Material and Shapes

- 'shaped' = the cross-section is formed to a tube, a box-section,...
- By 'efficient' we mean that, for given loading conditions, the section uses as little material, and is therefore as light, as possible.
- Tubes, boxes and I-sections will be referred to as 'simple shapes'.
- Even greater efficiencies are possible with sandwich panels (thin load-bearing skins bonded to a foam or honeycomb interior) and with structures (the Warren truss, for instance).
Section shape is important for certain modes of loading. When shape is a variable a new term, the shape factor $\phi$, appears in some of the material indices: they then allow optimum selection of material and shape.
How to choose, from among the vast range of materials and the section shapes in which they are available?

Shape factors: simple numbers which characterize the efficiency of shaped sections.

The best material-and-shape combination depends on the mode of loading.
Mode of loading and shape

- In axial tension, the area of the cross-section is important but its shape is not: all sections with the same area will carry the same load.
In bending: beams with hollow-box or I-sections are better than solid sections of the same cross-sectional area.
Torsion has its 'best' shapes: circular tubes, for instance, are better than either solid sections or I-sections.
Shape Factor

- A shape factor (symbol $\phi$) which measures, for each mode of loading, the efficiency of a shaped section.

- A shape factor is a dimensionless number which characterizes the efficiency of the shape, regardless of its scale, in a given mode of loading.
A material can be thought of as having properties but no shape; a **component** or a **structure** is a material made into a shape.

Mechanical efficiency is obtained by combining material with macroscopic shape. The shape is characterized by a dimensionless shape factor, $\phi$. The schematic is suggested by Parkhouse (1984).
A shape factor, $\phi_b^e$, for **elastic bending** of beams, and another, $\phi_t^e$ for elastic twisting of shafts (the superscript $e$ means elastic) = the appropriate shape factors when design is based on stiffness.

When based **on strength** (that is, on the first onset of plastic yielding or on fracture) two more shape factors are needed: $\phi_b^f$ and $\phi_t^f$ (the superscript $f$ means failure).

All four shape factors are defined so that they are equal to 1 for a **solid bar with a circular cross-section**.
Elastic bending and twisting

- Bending stiffness \((a\ \text{force per unit displacement})\) of a beam of length \(e\), made of a material with Young’s modulus \(E\), shear is negligible

\[
S \propto EI
\]

\[
I = \int_{\text{section}} y^2 \, dA
\]

- Shape enters through the second moment of area, \(I\), about the axis of bending
<table>
<thead>
<tr>
<th>Section shape</th>
<th>Area A ( (m) )</th>
<th>Moment I ( (m^4) )</th>
<th>Moment K ( (m^4) )</th>
<th>Moment Z ( (m^4) )</th>
<th>Moment Q ( (m^4) )</th>
<th>Moment Z(_p) ( (m^4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>( bh )</td>
<td>( \frac{bh^3}{12} )</td>
<td>( \frac{bh^3}{3} \left(1 - 0.58 \frac{b}{h}\right) )</td>
<td>( \frac{bh^2}{6} )</td>
<td>( \frac{b^2h^2}{(3h + 1.8b)} )</td>
<td>( \frac{bh^2}{4} )</td>
</tr>
<tr>
<td>Triangle</td>
<td>( \frac{\sqrt{3}}{4}a^2 )</td>
<td>( \frac{a^4}{32\sqrt{3}} )</td>
<td>( \frac{\sqrt{3}}{80}a^4 )</td>
<td>( \frac{a^3}{32} )</td>
<td>( \frac{a^3}{20} )</td>
<td>( \frac{3a^3}{64} )</td>
</tr>
<tr>
<td>Circle</td>
<td>( \pi r^2 )</td>
<td>( \frac{\pi r^4}{4} )</td>
<td>( \frac{\pi r^4}{2} )</td>
<td>( \frac{\pi r^4}{4} )</td>
<td>( \frac{\pi r^3}{2} )</td>
<td>( \frac{\pi r^3}{3} )</td>
</tr>
<tr>
<td>Ellipse</td>
<td>( \pi ab )</td>
<td>( \frac{\pi a^3b}{4(a^2 + b^2)} )</td>
<td>( \frac{\pi a^3b^3}{4a^2b} )</td>
<td>( \frac{\pi a^3b^3}{4a^2b} )</td>
<td>( \frac{\pi a^2b}{2} )</td>
<td>( \frac{\pi a^2b}{3} )</td>
</tr>
<tr>
<td>Cylinder</td>
<td>( \pi(r_o^2 - r_i^2) )</td>
<td>( \frac{\pi}{4}(r_o^4 - r_i^4) )</td>
<td>( \frac{\pi}{2}(r_o^4 - r_i^4) )</td>
<td>( \frac{\pi}{4r_o}(r_o^4 - r_i^4) )</td>
<td>( \frac{\pi}{2r_o}(r_o^4 - r_i^4) )</td>
<td>( \frac{\pi}{3}(r_o^3 - r_i^3) )</td>
</tr>
<tr>
<td>Rectangle 2b</td>
<td>( 2t(h + b) )</td>
<td>( \frac{1}{6}h^3(t + 3\frac{b}{h}) )</td>
<td>( \frac{2tb^2h^2}{(h + b)(1 - \frac{t}{h})^4} )</td>
<td>( \frac{1}{3}h^2(t + 3\frac{b}{h}) )</td>
<td>( \frac{2tbh}{3}(1 - \frac{t}{h})^2 )</td>
<td>( bht\left(1 + \frac{h}{2b}\right) )</td>
</tr>
</tbody>
</table>
\( I_o = \frac{b_o^4}{12} = \frac{A^2}{12} \)

\( I^0 \) for a square section of area \( A \) (Table 7.1)

\( \phi_B^e = \frac{S}{S_o} = \frac{EI}{EI_o} = \frac{12I}{A^2} \)

---

= dimensionless - \( I \) has dimensions of \((\text{length})^4\) and so does \( A^2 \).

- It depends only on shape: big and small beams have the same value of \( \phi_B^e \) if their section shapes are the same.
Solid equiaxed sections (circles, squares, hexagons, octagons) all have values very close to 1 - for practical purposes they can be set equal to 1.

But if the section is elongated, or hollow, or of I-section, or corrugated, things change:

A thin-walled tube or a slender I-beam can have a value of $\phi_B$ of 50 or more. Such a shape is efficient in that it uses less material (and thus less mass) to achieve the same bending stiffness*

A beam with $\phi_B = 50$ is 50 times stiffer than a solid beam of the same weight.
(a) A set of rectangular sections with $\phi_B^e = 2$.
(b) A set of I-sections with $\phi_B^e = 10$. (c) A set of tubes with $\phi_B^e = 15$. Members of a set differ in size but not in shape.
The effect of section shape on bending stiffness $EI$: a square-section beam compared: *left*, with a tube of the same area (but 2.5 times stiffer); *right*, with a tube with the same stiffness (but 4 times lighter).
The second moment of area $I$ plotted against section area $A$. Efficient structures have high values of the ratio $I/A^2$; inefficient structures (ones that bend easily) have low values. Real structural sections have values of $I$ and $A$ that lie in the shaded zones. Note that there are limits on $A$ and on the maximum shape efficiency $\phi_B^o$ that depend on material.
<table>
<thead>
<tr>
<th>Section shape</th>
<th>Bending factor, $\varphi_B^e$</th>
<th>Torsional factor, $\varphi_T^e$</th>
<th>Bending factor, $\varphi_B^f$</th>
<th>Torsional factor, $\varphi_T^f$</th>
<th>Bending factor, $\varphi_B^{pl}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{h}{b}$</td>
<td>$2.38 \frac{h}{b} \left(1 - 0.58 \frac{b}{h}\right)$</td>
<td>$\left(\frac{h}{b}\right)^{0.5}$</td>
<td>$1.6 \sqrt{\frac{b}{h}} \frac{1}{h(1 + 0.6b/h)}$</td>
<td>$\left(\frac{h}{b}\right)^{0.5}$</td>
</tr>
<tr>
<td></td>
<td>$(h &gt; b)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{2}{\sqrt{3}} = 1.15$</td>
<td>0.832</td>
<td>$\frac{3^{1/4}}{2} = 0.658$</td>
<td>0.83</td>
<td>$\frac{3^{1/4}}{2} = 0.658$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{\pi} = 0.955$</td>
<td>1.14</td>
<td>$\frac{3}{2\sqrt{\pi}} = 0.846$</td>
<td>1.35</td>
<td>$\frac{4}{3\sqrt{\pi}} = 0.752$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{3a}{\pi b}$</td>
<td>$\frac{2.28ab}{(a^2 + b^2)}$</td>
<td>$\frac{3}{2\sqrt{\pi}} \sqrt{\frac{a}{b}}$</td>
<td>1.35 $\sqrt{\frac{a}{b}}$ $(a &lt; b)$</td>
<td>$\frac{4}{3\sqrt{\pi}} \sqrt{\frac{a}{b}} = 0.752 \sqrt{\frac{a}{b}}$</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{\pi \left(\frac{r}{t}\right)}$</td>
<td>$1.14 \left(\frac{r}{t}\right)$</td>
<td>$\frac{3}{\sqrt{2\pi}} \sqrt{\frac{r}{t}}$</td>
<td>1.91 $\sqrt{\frac{r}{t}}$</td>
<td>$\sqrt{\frac{2}{\pi}} \sqrt{\frac{r}{t}}$</td>
</tr>
<tr>
<td></td>
<td>$(r \gg t)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2h} \left(1 + 3b/h\right)$</td>
<td>$\frac{3.57b^2(1-t/h)^4}{2t(1+b/h)^2}$</td>
<td>$\frac{1}{\sqrt{2}} \sqrt{\frac{h}{(1+b/h)^{3/2}}}$</td>
<td>3.39 $\sqrt{\frac{h^2}{bt}} \frac{1}{(1+b/h)^{3/2}}$</td>
<td>$\sqrt{2} \sqrt{\frac{h^2}{bt}} \left(1 + h/2b\right)$</td>
</tr>
<tr>
<td></td>
<td>$(h, b \gg t)$</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Shape</td>
<td>Expression</td>
<td>Shape</td>
<td>Expression</td>
<td>Shape</td>
<td>Expression</td>
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<td>-----------</td>
</tr>
<tr>
<td>3 ( a \left(1 + \frac{3b}{a}\right) ) ( \frac{3a}{\pi t \left(1 + \frac{b}{a}\right)^2} )</td>
<td>( \frac{3}{2\sqrt{\pi}} \frac{\sqrt{a}}{t \left(1 + \frac{b}{a}\right)^{3/2}} )</td>
<td>( 9.12(ab)^{5/2} )</td>
<td>( \frac{9.12a b^{5/2}}{\left(a^2 + b^2\right)^{3/2} \left(a + b\right)^2} )</td>
<td>( \frac{3}{2\sqrt{\pi}} \frac{\sqrt{a}}{t \left(1 + \frac{b}{a}\right)^{3/2}} )</td>
<td>( \frac{4}{\sqrt{\pi}} \frac{\sqrt{a^2}}{\sqrt{bt \left(1 + \frac{a}{b}\right)^{3/2}}} )</td>
</tr>
<tr>
<td>( \frac{3h_o^2}{2bt} ) ( h, b \gg t )</td>
<td>( \frac{3h_o}{\sqrt{2} \sqrt{bt}} )</td>
<td></td>
<td></td>
<td>( \sqrt{2} \frac{h_o}{\sqrt{bt}} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1h \left(1 + \frac{3b}{h}\right)}{2t \left(1 + \frac{b}{h}\right)^2} ) ( h, b \gg t )</td>
<td>( \frac{1.19 \left(\frac{t}{b}\right) \left(1 + \frac{4h}{b}\right)}{(1 + \frac{h}{b})^2} )</td>
<td>( \frac{1}{\sqrt{2}} \frac{\sqrt{h \left(1 + \frac{3b}{h}\right)}}{t \left(1 + \frac{b}{h}\right)^{3/2}} )</td>
<td>( \frac{1.13 \left(\frac{t}{b}\right) \left(1 + \frac{4h}{b}\right)}{(1 + \frac{h}{b})^2} )</td>
<td>( \frac{1}{\sqrt{2}} \frac{\sqrt{h^2 \left(1 + \frac{h}{2b}\right)}}{bt \left(1 + \frac{h}{b}\right)^{3/2}} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1h \left(1 + \frac{4b^2}{h^3}\right)}{2t \left(1 + \frac{b}{h}\right)^2} ) ( h, b \gg t )</td>
<td>( \frac{0.595 \left(\frac{t}{h}\right) \left(1 + \frac{8b}{h}\right)}{\left(1 + \frac{b}{h}\right)^2} )</td>
<td>( \frac{3}{4} \frac{\sqrt{h \left(1 + \frac{4b^2}{h^3}\right)}}{t \left(1 + \frac{b}{h}\right)^{3/2}} )</td>
<td>( \frac{0.565 \left(\frac{t}{h}\right) \left(1 + \frac{8b}{h}\right)}{h \left(1 + \frac{b}{h}\right)^{3/2}} )</td>
<td>( \frac{1}{\sqrt{2}} \frac{\sqrt{t \left(\frac{h}{(h + b)}\right)}}{\left[1 + \frac{2t(b - 2t)}{h^2}\right]^{3/2}} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1h \left(1 + \frac{4b^2}{h^3}\right)}{2t \left(1 + \frac{b}{h}\right)^2} ) ( h, b \gg t )</td>
<td>( \frac{1.19 \left(\frac{t}{h}\right) \left(1 + \frac{4b}{h}\right)}{(1 + \frac{b}{h})^2} )</td>
<td>( \frac{3}{4} \frac{\sqrt{h \left(1 + \frac{4b^2}{h^3}\right)}}{t \left(1 + \frac{b}{h}\right)^{3/2}} )</td>
<td>( \frac{1.13 \left(\frac{t}{h}\right) \left(1 + \frac{4b}{h}\right)}{h \left(1 + \frac{b}{h}\right)^{3/2}} )</td>
<td>( \frac{1}{\sqrt{2}} \frac{\sqrt{t \left(\frac{h}{(h + b)}\right)}}{\left[1 + \frac{2t(b - 2t)}{h^2}\right]^{3/2}} )</td>
<td></td>
</tr>
</tbody>
</table>
Shape factor for a shaft

- Shapes which resist bending well may not be so good when twisted.
- The stiffness of a shaft – the torque $T$ divided by the angle of twist $\theta$ (Figure 7.2(c)) - is given by

$$S_T = \frac{T}{\theta} = \frac{KG}{L}$$

where $L$ is length of the shaft and $G$ the shear modulus of the material of which it is made. Approximate expressions for $K$ are listed in Table 11.2.

The shape factor for elastic twisting is defined, as before, by the ratio of the torsional stiffness of the shaped section, $S_T$, to that, $S_{T_o}$, of a solid square shaft of the same length $L$ and cross-section $A$, which, using equation (11.5), is:
\[ \phi_T^e = \frac{S_T}{S_{T_0}} = \frac{K}{K_0} \]

The torsional constant \( K_0 \) for a solid square section (Table 11.1, top row with \( b = h \)) is

\[ K_0 = 0.14A^2 \]

giving

\[ \phi_T^e = 7.14 \frac{K}{A^2} \quad (11.7) \]
Failure in Bending

Failure in bending. Plasticity starts when the stress, somewhere, first reaches the yield strength $\sigma_y$. Fracture occurs when this stress first exceeds the fracture strength $\sigma_f$. Fatigue failure, if it exceeds the endurance limit $\sigma_e$. Any one of these constitutes failure. As in earlier chapters, we use the symbol $\sigma_f$ for the failure stress, meaning “the local stress that will first cause yielding or fracture or fatigue failure.”

In bending, the stress $\sigma$ is largest at the point $y_m$ on the surface of the beam that lies furthest from the neutral axis. Its value is

$$\sigma = \frac{M y_m}{I} = \frac{M}{Z} \tag{9.8}$$

where $M$ is the bending moment. Failure occurs when this stress first exceeds $\sigma_f$. Thus, in problems of beam failure, shape enters through the section modulus, $Z = I/y_m$. The strength efficiency of the shaped beam $\phi_B$ is measured by the ratio $Z/Z_0$, where $Z_0$ is the section modulus of a reference beam of square section with the same cross-sectional area, $A$: 
Thus,

\[ Z_\circ = \frac{b_0^3}{6} = \frac{A^{3/2}}{6} \]

\[ \phi_B = \frac{Z}{Z_\circ} = \frac{6Z}{A^{3/2}} \]

The section modulus $Z$ plotted against section area $A$. Efficient structures have high values of the ratio $Z/A^{3/2}$, inefficient structures (ones that bend easily) have low values. Real structural sections have values of $Z$ and $A$ that lie in the shaded zones. Note that there are limits on $A$ and on the maximum shape efficiency $\phi_B$ that depend on material.
### Definitions of shape factors

<table>
<thead>
<tr>
<th>Design constraint*</th>
<th>Bending</th>
<th>Torsion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stiffness</td>
<td>( \phi_B^C = \frac{12I}{A^2} )</td>
<td>( \phi_T^C = \frac{7.14K}{A^2} )</td>
</tr>
<tr>
<td>Strength</td>
<td>( \phi_B^f = \frac{6Z}{A^{3/2}} )</td>
<td>( \phi_T^f = \frac{4.8Q}{A^{3/2}} )</td>
</tr>
</tbody>
</table>
The efficiency of standard sections

- There are practical limits for the thinness of sections, and these determine, for a given material, the maximum attainable efficiency.

- These limits may be imposed by manufacturing constraints: the difficulty or expense of making an efficient shape may simply be too great.

- More often they are imposed by the properties of the material itself because these determine the failure mode of the section.
We explore the ultimate limits for shape efficiency. This we do in two ways.

The first is empirical: by examining the shapes in which real materials - steel, aluminium, etc. - are actually made, recording the limiting efficiency of available sections.

The second is by the analysis of the mechanical stability of shaped sections.
Standard sections for beams, shafts, and columns are generally **prismatic**.

Prismatic shapes are easily made by rolling, extrusion, drawing, pultrusion or sawing.

Figure 7.5 shows the taxonomy of the kingdom of prismatic shapes.

The section may be solid, closed-hollow (like a tube or box) or open-hollow (an I-, U- or L-section, for instance).
Each class of shape can be made in a range of materials. Those for which standard, off-the-shelf.

Sections are available are listed on the figure: steel, aluminium, GFRP and wood.

Each section has a set of attributes: they are the parameters used in structural or mechanical design. They include its dimensions and its section properties (the ‘moments’ I, \( K \) and the ‘section moduli’ Z and Q)
Fig. 7.5 A taxonomy of prismatic shapes, illustrating the attributes of a shaped section.
Figure 7.6(a). It shows \( \log(I) \) plotted against \( \log(A) \). Taking logarithms of the equation for the first shape factor \( \phi_B^o = 4\pi I/A^2 \) gives, after rearrangement,

\[
\log I = 2 \log A + \log \frac{\phi_B^o}{4\pi}
\]
Steel

$Z_{\text{max}}$ vs. $A$

Section Modulus for Bending, $Z_{\text{max}}$ ($10^{-10} \text{ m}^3$)

Shape Factor $\phi_B = 12$

Shape Factor $\phi_B = 1$

Data: as for I-A plot for Steel

Section Area, $A$ ($10^{-6} \text{ m}^2$)
Steel

$K$ vs. $A$

Shape Factor $\phi_T = 25$

Shape Factor $\phi_T = 1$

Data: as for I-A plot for Steel
Upper Limit for Shape Factor

### Empirical upper limits for the shape factors $\phi^e_B, \phi^e_T, \phi^f_B$ and $\phi^f_T$

<table>
<thead>
<tr>
<th>Material</th>
<th>$(\phi^e_B)_{max}$</th>
<th>$(\phi^e_T)_{max}$</th>
<th>$(\phi^f_B)_{max}$</th>
<th>$(\phi^f_T)_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structural steel</td>
<td>65</td>
<td>25</td>
<td>13</td>
<td>7</td>
</tr>
<tr>
<td>6061 aluminum alloy</td>
<td>44</td>
<td>31</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>GFRP and CFRP</td>
<td>39</td>
<td>26</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>Polymers (e.g. nylons)</td>
<td>12</td>
<td>8</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Woods (solid sections)</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Elastomers</td>
<td>$&lt; 6$</td>
<td>3</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
Upper Limit for Shape Factor
■ the *upper-limiting shape factor for simple shapes depends on material.*

■ The upper limits for shape efficiency are important.

■ They are central to the design of lightweight structures, and structures in which, for other reasons (cost, perhaps) the material content should be minimized.
Material limits for shape factors

- The range of shape factor for a given material is limited either by manufacturing constraints, or by local buckling.
- Steel, for example, can be drawn to thin-walled tubes or formed (by rolling, folding or welding) into efficient I-sections; shape factors as high as 50 are common.
Material limits for shape factors

- Wood cannot so easily be shaped; ply-wood technology could, in principle, be used to make thin tubes or I-sections, but in practice, shapes with values of $\phi$ greater than 5 are uncommon. That is a manufacturing constraint.

- Composites, too, can be limited by the present difficulty in making them into thin-walled shapes, although the technology for doing this now exists.
Material indices which include shape

- **Axial tension of ties**
  - The ability of a tie to carry a load $F$ without deflecting excessively or failing depends only on the area of its section, but **not on its shape**.
  - The material index for stiffness at minimum weight, $E/p$, holds for all section shapes.
  - **This is not true of bending or twisting, or when columns buckle.**
Elastic bending of beams and twisting of shafts

- Consider the selection of a material for a beam of specified stiffness $S_B$ and length $e$, and it is to have minimum mass, $m$.
- The selection must allow for the fact that the available candidate materials have section shapes which differ.
The mass \( m \) of a beam of length \( l \) and section area \( A \) is

\[
m = A\ell \rho \quad \text{(7.22)}
\]

Its bending stiffness is given by

\[
S_B = C_1 \frac{EI}{L^3}
\]

Replacing \( I \) by \( \phi_B^e \) using

\[
\phi_B^e = \frac{S}{S_o} = \frac{EI}{EI_o} = \frac{12I}{A^2}
\]
a material-shape combination for a light stiff beam, the best choice is that with the greatest value of the index

\[ S_B = \frac{C_1}{12} \frac{E}{L^3} \phi_B A^2 \]

\[ m = \left( \frac{12S_B}{C_1} \right)^{1/2} L^{5/2} \left[ \frac{\rho}{(\phi_B^E)^{1/2}} \right] \]

\[ M_1 = \frac{(\phi_B^E)^{1/2}}{\rho} \]
Elastic twisting of shafts

The procedure for elastic twisting of shafts is similar. A shaft of section $A$ and length $l$ is subjected to a torque $T$. It twists through an angle $\theta$.

It is required that the torsional stiffness, $T/\theta$, meet a specified target $S_T$, at minimum mass. The mass of the shaft is given, as before, by

$$m = A\ell \rho$$  \hspace{1cm} (7.22)
Its torsional stiffness is

\[ S_T = \frac{KG}{\phi_T} \]

\[ \phi_T = 7.14 \frac{K}{A^2} \]

\[ S_T = \frac{G}{7.14L} \phi_T A^2 \]
Using this to eliminate $A$

$$m = \left( 7.14 \frac{S_T}{L^3} \right)^{1/2} L^{3/2} \left[ \frac{\rho}{(\phi_1 G)^{1/2}} \right]$$

The best material-and-shape combination is that with the greatest value of $(\phi_1^c G)/\rho^{1/2}$. The shear modulus, $G$, is closely related to Young’s modulus $E$. For the practical purposes we approximate $G$ by $3/8E$; when the index becomes

$$M_2 = \frac{(\phi_1^c E)^{1/2}}{\rho} \quad (11.32)$$

For shafts of the same shape, this reduces to $E^{1/2}/\rho$ again. When shafts differ in both material and shape, the material index (equation (11.32)) is the one to use.
Failure of beams and shafts

- A beam, loaded in bending, must support a specified load $F$ without failing. The mass of the beam is to be minimized.
- When section-shape is a variable, the best choice is found as follows.
- Failure occurs if the load exceeds the failure moment

$$M = Z\sigma_f$$
Replacing $Z$ by the appropriate shape-factor $\phi_B^f$ via

$$\phi_B^f = \frac{Z}{Z_0} = \frac{6Z}{A^{3/2}}$$

$$M = \frac{\sigma_f}{6} \phi_B^f A^{3/2}$$

Substituting this into equation (11.24) for the mass of the beam gives

$$m = (6M)^{2/3} L \left[ \frac{\rho^{3/2}}{\phi_B^f \sigma_f} \right]^{2/3}$$

(11.34)

The best material-and-shape combination is that with the greatest value of the index

$$M_3 = \frac{(\phi_B^f \sigma_f)^{2/3}}{\rho}$$

(11.35)
The twisting of shafts

- The twisting of shafts is treated in the same way. A shaft must carry a torque $T$ without failing. This requires that $T$ not exceed the failure torque $T_f$, from

$$T_f = Q \tau_f$$

Replacing $Q$ by $\phi_T^f$ with equation

$$\phi_T^f = \frac{Q}{Q_o} = 4.8 \frac{Q}{A^{3/2}}$$
\[ T_f = \frac{\sigma_f}{4\sqrt{\pi}} \phi_T A^{3/2} \quad (7.33) \]

where \( \tau_f \), the shear-failure strength has been replaced by \( \sigma_f/2 \), the tensile failure strength. Using this to eliminate the area \( A \) in equation (7.34) for the mass of the shaft gives

\[ m = \left( 4\sqrt{\pi} \frac{T_f}{\ell^3} \right)^{2/3} \ell^3 \left[ \frac{\rho^{3/2}}{\phi_T \sigma_f} \right]^{2/3} \quad (7.34) \]

Performance is maximized by the selection which has the greatest value of

\[ M_4 = \frac{(\phi_T^f \sigma_f)^{2/3}}{\rho} \quad (7.35) \]
The microscopic or micro-structural shape factor

Example: wood.

- The solid component of wood (a composite of cellulose, lignin and other polymers) is shaped into little prismatic cells, dispersing the solid further from the axis of bending or twisting of the branch or trunk of the tree.

- This gives wood a greater bending and torsional stiffness than the solid of which it is made.
The added efficiency is characterized by a set of *microscopic shape factors*, \( \varphi \), with definitions and characteristics exactly like those of \( \theta \).

The characteristic of microscopic shape is that the structure repeats itself; it is *extensive*.
Mechanical efficiency can be obtained by combining material with microscopic, or internal, shape, which repeats itself to give an extensive structure. The shape is characterized by microscopic shape factors, $\psi$. 
Many natural materials have microscopic shape. Wood is just one example.
Four extensive microstructured materials that are mechanically efficient: (a) prismatic cells, (b) fibers embedded in a foamed matrix, (c) concentric cylindrical shells with foam between, and (d) parallel plates separated by foamed spacers.
- A wood-like structure = hexagonal-prismatic cells; it has translational symmetry and is uniform, with isotropic properties in the plane of the section when the cells are regular hexagons.

- An array of fibres separated by a foamed matrix typical of palm wood; it too is uniform in-plane and has translational symmetry.

- An axisymmetric structure of concentric cylindrical shells separated by a foamed matrix, like the stem of some plants. And the fourth is a layered structure, a sort of multiple sandwich-panel, like the shell of the cuttle fish; it has orthotropic symmetry.
Microscopic shape factors

Consider the gain in bending stiffness when a solid cylindrical beam like that shown as a black circle in Figure 7.11 is expanded, at constant mass, to a circular beam with any one of the structures which surround it in the figure. The stiffness $S_s$ of the original solid beam is

$$S_s = \frac{C_1 E_s I_s}{\ell^3}$$  \hspace{1cm} (7.36)

where the subscript $s$ means a property of the solid beam. When the beam is expanded at constant mass its density falls from $\rho_s$ to $\rho$ and its radius increases from $r_s$ to

$$r = \left( \frac{\rho_s}{\rho} \right)^{1/2} r_s$$  \hspace{1cm} (7.37)

with the result that its second moment of area increases from $I_s$ to

$$I = \frac{\pi}{4} r^4 = \frac{\pi}{4} \left( \frac{\rho_s}{\rho} \right)^2 r_s^4 = \left( \frac{\rho_s}{\rho} \right)^2 I_s$$  \hspace{1cm} (7.38)

If the cells, fibres, rings or plates in Figure 7.11 are extensive parallel to the axis of the beam, the modulus falls from that of the solid, $E_s$, to

$$E = \left( \frac{\rho}{\rho_s} \right) E_s$$  \hspace{1cm} (7.39)
The stiffness of the expanded beam is thus

\[ S = \frac{C_1 EI}{\ell^3} = \frac{C_1 E_s I_s}{\ell^3} \left( \frac{\rho_s}{\rho} \right) \]  

(7.40)

The microscopic shape factor, \( \psi \) is defined in the same way as the macroscopic one, \( \phi \): it is the ratio of the stiffness of the structured beam to that of the solid one. Taking the ratio of equations (7.40) and (7.36) gives

\[ \psi^e_B = \frac{S}{S_s} = \frac{\rho_s}{\rho} \]  

(7.41)

In words: the microscopic shape factor for prismatic structures is simply the reciprocal of the relative density. Note that, in the limit of a solid (when \( \rho^* = \rho_s \)) \( \psi^e_B \) takes the value 1, as it obviously should.
A similar analysis for failure in bending gives the shape factor

$$\psi_B^f = \left( \frac{\rho_s}{\rho} \right)^{1/2}$$  (7.42)

Torsion, as always, is more difficult. When the structure of Figure 7.11(c), which has circular symmetry, is twisted, its rings act like concentric tubes and for these

$$\psi_T^c = \frac{\rho_s}{\rho} \text{ and } \psi_T^f = \left( \frac{\rho_s}{\rho} \right)^{1/2}$$  (7.43)
Structuring, then, converts a solid with modulus \( E_s \) and strength \( \sigma_{f,s} \) to a new solid with properties \( E \) and \( \sigma_f \). If this new solid is formed to an efficient macroscopic shape (a tube, say, or an I-section) its bending stiffness, to take an example, increases by a further factor of \( \phi_B^e \). Then the stiffness of the beam, expressed in terms of that of the solid of which it is made, is

\[
S = \psi_B^e \phi_B^e S_s
\]  

(7.44)

that is, the shape factors multiply. The same is true for strength.
Co-selecting material and shape

- Co-selection by calculation
- Graphical co-selection using material property charts
Co-selection by calculation

Consider as an example the selection of a material for a stiff shaped beam of minimum mass. Four materials are available, listed in Table 7.4 with their properties and the shapes, characterized by $\phi^e_B$, in which they are available (here, the maximum ones). We want the combination with the largest value of the index $M_1$ of equation (7.27) which, repeated, is

$$M_1 = \frac{(\phi^e_B E)^{1/2}}{\rho}$$

The second last column shows the simple ‘fixed shape’ index $E^{1/2}/\rho$: wood has the greatest value — it is more than twice as stiff as steel for the same weight. But when each material is shaped efficiently (last column) wood has the lowest value of $M_1$ — even steel is better; the aluminium alloy wins, marginally better than GFRP.
Efficiency gain through shape

A shaped material is required for a stiff beam of minimum mass. Four materials are available with the properties and typical shapes listed in Table 9.5. Which material-shape combination has the lowest mass for a given stiffness?

Answer
The second to last column of the table shows the simple “fixed shape” index $E^{1/2}/\rho$: Wood has the greatest value, more than twice that of steel. But when each material is shaped efficiently (last column), wood has the lowest value of $M_1$—even steel is better; the aluminum alloy wins, surpassing steel and GFRP.

<table>
<thead>
<tr>
<th>Material</th>
<th>$\rho$ (Mg/m$^3$)</th>
<th>$E$ (GPa)</th>
<th>$\phi_B^o$</th>
<th>$E^{1/2}/\rho$</th>
<th>$(\phi_B^o E)^{1/2}/\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1020 steel</td>
<td>7.85</td>
<td>205</td>
<td>20</td>
<td>1.8</td>
<td>8.2</td>
</tr>
<tr>
<td>6061-T4 Al</td>
<td>2.7</td>
<td>70</td>
<td>15</td>
<td>3.1</td>
<td>12.0</td>
</tr>
<tr>
<td>GFRP (isotropic)</td>
<td>1.75</td>
<td>28</td>
<td>8</td>
<td>2.9</td>
<td>8.5</td>
</tr>
<tr>
<td>Wood (oak)</td>
<td>0.9</td>
<td>13.5</td>
<td>2</td>
<td><strong>4.1</strong></td>
<td>5.8</td>
</tr>
</tbody>
</table>
Choosing material and shape combinations

A shaped material is required for a strong beam of minimum mass. Four materials are available with the properties and shapes listed in Table 9.6. Which combination has the lowest mass for a given bending strength?

Answer

The second to last column of the table shows the simple “fixed-shape” index $\sigma_t^{2/3}/\rho$: Wood has the greatest value, more than three times that of steel. But when each material is shaped (last column) the aluminum alloy wins, surpassing steel and GFRP.

<table>
<thead>
<tr>
<th>Material</th>
<th>$\rho$ (Mg/m$^3$)</th>
<th>$\sigma_t$ (MPa)</th>
<th>$\phi_B^t$</th>
<th>$\sigma_t^{2/3}/\rho$</th>
<th>$(\phi_B^t\sigma_t)^{2/3}/\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1020 steel, normalized</td>
<td>7.85</td>
<td>330</td>
<td>5</td>
<td>6.1</td>
<td>17.8</td>
</tr>
<tr>
<td>6061-T4 Al</td>
<td>2.7</td>
<td>110</td>
<td>4</td>
<td>8.5</td>
<td><strong>21.4</strong></td>
</tr>
<tr>
<td>GFRP SMC (isotropic)</td>
<td>2.0</td>
<td>80</td>
<td>3</td>
<td>9.3</td>
<td>19.3</td>
</tr>
<tr>
<td>Wood (oak), along grain</td>
<td>0.9</td>
<td>50</td>
<td>1.5</td>
<td><strong>15</strong></td>
<td>19.7</td>
</tr>
</tbody>
</table>
Graphical co-selection using material property charts

Shaped materials can be displayed and selected with the Material Selection Charts. The reasoning, for the case of elastic bending, goes like this. The material index for elastic bending (equation (7.27)) can be rewritten as

$$M_1 = \frac{(\phi_B^e E)^{1/2}}{\rho} = \frac{(E/\phi_B^e)^{1/2}}{\rho/\phi_B^e}$$  \hspace{1cm} (7.45)

The equation says: a material with modulus $E$ and density $\rho$, when structured, behaves like a material with modulus

$$E^* = E/\phi_B^e$$

and density

$$\rho^* = \rho/\phi_B^e$$
The structured material behaves like a new material with a modulus $E^* = E/\phi_B^\circ$ and a density $\rho^* = \rho/\phi_B^\circ$, moving it from a position below the broken selection line to one above. A similar procedure can be applied for bending strength, as described in the text.
Materials selection based on strength (rather than stiffness) at a minimum weight uses the chart of strength $\sigma_f$ against density $\rho$, shown schematically in Figure 7.13. Shape is introduced in a similar way. The material index for failure in bending (equation (7.32)), can be rewritten as follows

$$M_3 = \frac{(\phi_B^f \sigma_f)^{2/3}}{\rho} = \frac{(\sigma_f/(\phi_B^f)^2)^{2/3}}{\rho/(\phi_B^f)^2}$$  

(7.32)

The material with strength $\sigma_f$ and density $\rho$, when shaped, behaves in bending like a material of strength

$$\sigma_f^* = \frac{\sigma_f}{(\phi_B^f)^2}$$

and density

$$\rho^* = \frac{\rho}{(\phi_B^f)^2}$$
Fig. 7.13 Schematic of Materials Selection Chart 2: strength $\sigma_f$ plotted against density $\rho$. The best material for a light, strong beam is that with the greatest value of $\sigma_f^{2/3}/\rho$. The structured material behaves in bending like a new material with strength $\sigma_f' = \sigma_f/\phi^2$, and density $\rho/\phi^2$ (where $\phi$ means $\phi_B$), and can be plotted onto the chart. All the material-selection criteria still apply. A similar procedure is used for torsional strength.