Elements of the theory of plasticity

Subjects of interest

- Introduction/objectives
- The flow curve
- True stress and true strain
- Yielding criteria for ductile materials
- Combined stress tests
- The yield locus
- Anisotropy in yielding
Objective

• This chapter provides a basic theory of plasticity for the understanding of the flow curve.

• Differences between the true stress – true strain curve and the engineering stress – engineering strain curves will be highlighted.

• Finally the understanding of the yielding criteria for ductile materials will be made.
Introduciton

- Plastic deformation is a non reversible process where Hooke’s law is no longer valid.

- One aspect of plasticity in the viewpoint of structural design is that it is concerned with predicting the maximum load, which can be applied to a body without causing excessive yielding.

- Another aspect of plasticity is about the plastic forming of metals where large plastic deformation is required to change metals into desired shapes.
The flow curve

• True stress-strain curve for typical ductile materials, i.e., aluminium, show that the stress-strain relationship follows up the *Hooke’s law* up to the *yield point*, \( \sigma_0 \).

• Beyond \( \sigma_0 \), the metal *deforms plastically* with *strain-hardening*. This cannot be related by any simple constant of proportionality.

• If the load is released from straining up to point \( A \), the *total strain* will immediately decrease from \( \varepsilon_1 \) to \( \varepsilon_2 \) by an amount of \( \sigma/E \).

• The strain \( \varepsilon_1-\varepsilon_2 \) is the *recoverable elastic strain*. Also there will be a small amount of the plastic strain \( \varepsilon_2-\varepsilon_3 \) known as *anelastic behaviour* which will disappear by time.\( \rightarrow \) (neglected in plasticity theories.)
The flow curve

• Usually the stress-strain curve on unloading from a plastic strain will not be exactly linear and parallel to the elastic portion of the curve.
• On reloading the curve will generally bend over as the stress pass through the original value from which it was unloaded.
• With this little effect of unloading and loading from a plastic strain, the stress-strain curve becomes a continuation of the hysteresis behaviour (but generally neglected in plasticity theories.)
The flow curve

- If specimen is deformed plastically beyond the yield stress in tension (+), and then in compression (-), it is found that the yield stress on reloading in compression is less than the original yield stress.

- The dependence of the yield stress on loading path and direction is called the Bauschinger effect. (however it is neglected in plasticity theories and it is assumed that the yield stress in tension and compression are the same).
The flow curve

• A true stress – strain curve provides the stress required to cause the metal to flow plastically at any strain → it is often called a ‘flow curve’.

• A mathematical equation that fit to this curve from the beginning of the plastic flow to the maximum load before necking is a power expression of the type

\[ \sigma = K \varepsilon^n \]  

...Eq.1

Where \( K \) is the stress at \( \varepsilon = 1.0 \)
\( n \) is the strain – hardening exponent
(slope of a log-log plot of Eq.1)

Note: higher \( \sigma_0 \) → greater elastic region, but less ductility (less plastic region).

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Due to considerable mathematical complexity concerning the theory of plasticity, the idealised flow curves are therefore utilised to simplify the mathematics.

1) **Rigid ideal plastic material**: no elastic strain, no strain hardening.
2) **Perfectly plastic material with an elastic region**, i.e., plain carbon steel.
3) **Piecewise linear (strain-hardening material)**: with elastic region and strain hardening region → more realistic approach but complicated mathematics.
True stress and true strain

- The *engineering stress – strain curve* is based entirely on the *original dimensions* of the specimen → This cannot represent true deformation characteristic of the material.

- The *true stress – strain curve* is based on the *instantaneous specimen dimensions*.
The true strain

According to the concept of unit linear strain,

\[ e = \frac{\Delta L}{L_o} = \frac{1}{L_o} \int_{L_o}^L dL \] \hspace{2cm} \text{...Eq.2}

This satisfies for elastic strain where \( \Delta L \) is very small, but not for plastic strain.

Definition: true strain or natural strain (first proposed by Ludwik) is the change in length referred to the instantaneous gauge length.

\[ \varepsilon = \sum \frac{L_i - L_o}{L_o} + \frac{L_2 - L_1}{L_1} + \frac{L_3 - L_2}{L_2} + \ldots \]

\[ \varepsilon = \int_{L_o}^L \frac{dL}{L} = \ln \frac{L}{L_o} \] \hspace{2cm} \text{...Eq.3}

Hence the relationship between the true strain and the conventional linear strain becomes

\[ e = \frac{\Delta L}{L_o} = \frac{L - L_o}{L_o} = \frac{L}{L_o} - 1 \]

\[ e + 1 = \frac{L}{L_o} \]

\[ \varepsilon = \ln \frac{L}{L_o} = \ln(e + 1) \] \hspace{2cm} \text{...Eq.4}
Comparison of true strain and conventional linear strain

<table>
<thead>
<tr>
<th>True stain $\varepsilon$</th>
<th>0.01</th>
<th>0.10</th>
<th>0.20</th>
<th>0.50</th>
<th>1.0</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional strain $\varepsilon$</td>
<td>0.01</td>
<td>0.105</td>
<td>0.22</td>
<td>0.65</td>
<td>1.72</td>
<td>53.6</td>
</tr>
</tbody>
</table>

• In true strain, the same amount of strain (but the opposite sign) is produced in tension and compression respectively.

**Ex:** Expanding the cylinder to twice its length.

**Tension**

$$\varepsilon = \ln\left(\frac{2L_f}{L_o}\right) = \ln 2$$

...Eq.5

**Ex:** Compression to half the original length.

**Compression**

$$\varepsilon = \ln\left[\frac{(L_o/2)L_o}\right] = -\ln 2$$

...Eq.6
Total true strain and conventional strain

<table>
<thead>
<tr>
<th>Increment</th>
<th>Length of rod</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>1</td>
<td>55</td>
</tr>
<tr>
<td>2</td>
<td>60.5</td>
</tr>
<tr>
<td>3</td>
<td>66.5</td>
</tr>
</tbody>
</table>

The total conventional strain $e$

- The total conventional strain $e_{0-3}$ is not equal to $e_{0-1} + e_{1-2} + e_{2-3}$.

$$e_{0-1} + e_{1-2} + e_{2-3} = 0.3 \neq e_{0-3} = \frac{16.55}{50} = 0.331 \quad \text{...Eq.7}$$

The total true strain $\varepsilon$

- The total true strain = the summation of the incremental true strains.

$$\varepsilon_{0-1} + \varepsilon_{1-2} + \varepsilon_{2-3} = \ln \frac{55}{50} + \ln \frac{60.5}{55} + \ln \frac{66.55}{60.5} = \ln \frac{66.55}{50} = \varepsilon_{0-3} = 0.286 \quad \text{...Eq.8}$$
The volume strain

According to the volume strain $\Delta$

$$\Delta = \frac{\Delta V}{V} = \frac{(1 + e_x)(1 + e_y)(1 + e_z)dx\,dy\,dz - dx\,dy\,dz}{dx\,dy\,dz}$$

$$\Delta = (1 + e_x)(1 + e_y)(1 + e_z) - 1$$ …Eq.9

During plastic deformation, it is considered that the volume of a solid remain constant $\rightarrow (\Delta = 0)$

$$\Delta + 1 = (1 + e_x)(1 + e_y)(1 + e_z)$$

$$\ln 1 = 0 = \ln(1 + e_x) + \ln(1 + e_y) + \ln(1 + e_z)$$

But $\varepsilon_x = \ln(1+e_x)$, hence

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$$ …Eq.10

Due to the constant volume $A_oL_o = AL$, therefore

$$\varepsilon = \ln \frac{L}{L_o} = \ln \frac{A_o}{A}$$ …Eq.11
**The true stress**

**Definition:** *the true stress* is the load divided by the instantaneous area.

True stress \[ \sigma = \frac{P}{A} \]

Engineering stress \[ s = \frac{P}{A_o} \]

Relationship between the true stress and the engineering stress

Since
\[ \sigma = \frac{P}{A} = \frac{P}{A_o} \times \frac{A_o}{A} \]

But
\[ \frac{A_o}{A} = \frac{L}{L_o} = e + 1 \]

Hence,
\[ \sigma = \frac{P}{A_o} (e + 1) = s(e + 1) \]

...Eq.12
**Example:** A tensile specimen with a 12 mm initial diameter and 50 mm gauge length reaches maximum load at 90 kN and fractures at 70 kN. The maximum diameter at fracture is 10 mm. Determine engineering stress at maximum load (the ultimate tensile strength), true fracture stress, true strain at fracture and engineering strain at fracture.

**Engineering stress at maximum load**

\[
\frac{P_{\text{max}}}{A_{\text{max}}} = \frac{90 \times 10^3}{\pi (12 \times 10^{-3})^2 / 4} = 796 \text{ MPa}
\]

**True fracture stress**

\[
\frac{P_f}{A_f} = \frac{70 \times 10^3}{\pi (10 \times 10^{-3}) / 4} = 891 \text{ MPa}
\]

**True strain at fracture**

\[
\varepsilon_f = \ln \frac{A_o}{A_f} = \ln \left( \frac{12}{10} \right)^2 = 2 \ln 1.2 = 0.365
\]

**Engineering strain at fracture**

\[
e_f = \exp(\varepsilon) - 1 = \exp(0.365) - 1 = 0.44
\]
Yielding criteria for ductile metals

• Plastic yielding of the material subjected to any external forces is of considerable importance in the field of plasticity.

• For predicting the onset of yielding in ductile material, there are at present two generally accepted criteria,

1) Von Mises’ or Distortion-energy criterion

2) Tresca or Maximum shear stress criterion
**Von Mises’ criterion**

*Von Mises* proposed that yielding occur when the second invariant of the stress deviator $J_2 > \text{critical value } k^2$.

$$\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = 6k^2$$  \hspace{1cm} \text{...Eq.13}

For yielding in uniaxial tension, $\sigma_1 = \sigma_o$, $\sigma_2 = \sigma_3 = 0$

$$2\sigma_o^2 = 6k^2, \text{then } k = \frac{\sigma_o}{\sqrt{3}}$$  \hspace{1cm} \text{...Eq.14}

Substituting $k$ from Eq.14 in Eq.13, we then have the von Mises’ yield criterion

$$\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{\frac{1}{2}} = \sqrt{2}\sigma_o$$  \hspace{1cm} \text{...Eq.15}

In pure shear, to evaluate the constant $k$, note $\sigma_1 = \sigma_3 = \tau_y$, $\sigma_2 = 0$, where $\sigma_o$ is the yield stress; when yields: $\tau_y^2 + \tau_y^2 + 4\tau_y^2 = 6k^2$ then $k = \tau_y$

By comparing with Eq 14, we then have

$$\tau_y = 0.577\sigma_o$$  \hspace{1cm} ***  \text{...Eq.16}
**Example:** Stress analysis of a spacecraft structural member gives the state of stress shown below. If the part is made from 7075-T6 aluminium alloy with $\sigma_0 = 500$ MPa, will it exhibit yielding? If not, what is the safety factor?

From *Eq. 16*

\[
\sigma_0 = \frac{1}{\sqrt{2}} \left( (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2) \right)^{1/2}
\]

\[
\sigma_0 = \frac{1}{\sqrt{2}} \left[ (200 - 100)^2 + (100 - (-50))^2 + (-50 - 200)^2 + 6(30)^2 \right]^{1/2}
\]

\[
\sigma_0 = 224 \text{ MPa}
\]

The calculated $\sigma_0 = 224$ MPa < the yield stress (500 MPa), therefore yielding will not occur.

**Safety factor** = $500/224 = 2.2$. 
Yielding occurs when the \textit{maximum shear stress} $\tau_{\text{max}}$ reaches the value of the shear stress in the uniaxial-tension test, $\tau_o$.

Where $\sigma_1$ is the algebraically largest and $\sigma_3$ is the algebraically smallest principal stress.

For uniaxial tension, $\sigma_1 = \sigma_o$, $\sigma_2 = \sigma_3 = 0$, and the shearing yield stress $\tau_o = \sigma_o/2$.

Therefore the maximum - shear stress criterion is given by

\[ \tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} \] \hspace{1cm} \text{...Eq.17}

\[ \tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} = \tau_o = \frac{\sigma_o}{2} \] \hspace{1cm} \text{...Eq.18}

\[ \sigma_1 - \sigma_3 = \sigma_o \] \hspace{1cm} \text{...Eq.19}

In pure shear, $\sigma_1 = -\sigma_3 = k$, $\sigma_2 = 0$, $\tau_{\text{max}} = \tau_y$

\[ \tau_y = 0.5\sigma_o \] \hspace{1cm} \text{***} \hspace{1cm} \text{...Eq.20}
**Example:** Use the maximum-shear-stress criterion to establish whether yielding will occur for the stress state shown in the previous example.

The calculated value of $\sigma_o$ is less than the yield stress (500 MPa), therefore yielding will not occur.
1) **Von Mises’ yield criterion**

- Yielding is based on differences of normal stress, but independent of hydrostatic stress.
- Complicated mathematical equations.
- Used in most theoretical work.

\[ \tau_y = 0.577\sigma_o \]

2) **Tresca yield criterion**

- Less complicated mathematical equation
- \( \Rightarrow \) used in engineering design.

\[ \tau_y = 0.5\sigma_o \]

*Note: the difference between the two criteria are approximately 1-15%.*
Combined stress tests

In a thin-wall tube, states of stress are various combinations of uniaxial and torsion with maybe a hydrostatic pressure being introduced to produce a circumferential hoop stress in the tube.

In a thin wall, $\sigma_1 = -\sigma_3, \sigma_2 = 0$

The maximum shear-stress criterion of yielding in the thin wall tube is given by

$$\left( \frac{\sigma_x}{\sigma_o} \right)^2 + 4 \left( \frac{\tau_{xy}}{\sigma_o} \right)^2 = 1$$

...Eq. 21

The distortion-energy theory of yielding is expressed by

$$\left( \frac{\sigma_x}{\sigma_o} \right)^2 + 3 \left( \frac{\tau_{xy}}{\sigma_o} \right)^2 = 1$$

...Eq. 22

Comparison between maximum-shear-stress theory and distortion-energy (von Mise’s) theory.
The yield locus

For a \textit{biaxial plane-stress} condition ($\sigma_2 = 0$) the \textit{von-Mise’s yield criterion} can be expressed as

$$\sigma_1^2 + \sigma_3^2 - \sigma_1 \sigma_3 = \sigma_0^2$$

...Eq.23

The equation is an ellipse type with

- major semiaxis $\sqrt{2}\sigma_0$ - minor semiaxis $\sqrt{\frac{2}{3}}\sigma_0$

\begin{itemize}
  \item The \textit{yield locus} for the \textit{maximum shear stress criterion} falls inside the \textit{von Mise’s yield ellipse}.
  \item The yield stress predicted by the \textit{von Mise’s criterion} is 15.5\% > than the yield stress predicted by the \textit{maximum-shear-stress criterion}.
\end{itemize}
References