Stress and strain relationships for elastic behaviour

Subjects of interest

• Introduction/Objectives
• Description of stress at a point
• State of stresses in two dimensions (Plane stress)
• Mohr’s circle of stress – two dimensions
• State of stress in three dimensions
• Strain at a point
• Hydrostatic and deviator components of stress
• Elastic stress-strain relations
• Strain energy
• Stress concentration
• Finite element method
Objectives

• This chapter provides mathematical relationships to understand relationships between stress and strain in a solid which obeys Hook’s law.

• Stress and strain at a point and in three dimensions will be understood.
Introduction

Solid

Elastic behaviour
Plastic behaviour

Stress-strain relationship

Two dimensional
Three dimensional

Note: if solid is metal, we need to include metallurgical factors.
Description of stress at a point

- **Stress** at a point is resolved into normal and shear components.
- **Shear components** are at arbitrary angles to the coordinate axes.

Stress acting on an element cube.

- The **normal stress** $\sigma_x$ acting on the plane perpendicular to the $x$ direction. (this also applies to $\sigma_y$ and $\sigma_z$.)

- The **shearing stress** has two components and need two subscripts;
  - The **first subscript** is the plane in which the stress acts.
  - The **second subscript** is the direction in which the stress acts.

**Ex:** $\tau_{yz}$ is the **shear stress** in the plane perpendicular to the $y$ axis in the $z$ direction.
**Sign convention for shear stress**

- A shear stress is **positive** if it points in the **positive** direction on the **positive** face of a unit cube. (and negative direction on the negative face).
- A shear stress is **negative** if it points in the **negative** direction of a **positive** face of a unit cube. (and positive direction on the negative face).
Stress components

• In order to establish state of stress at a point, nine quantities must be defined; $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$, $\tau_{xz}$, $\tau_{yx}$, $\tau_{yx}$, $\tau_{zx}$ and $\tau_{zy}$.

• If stress are slowly varying across the infinitesimal cube, moment equilibrium about the centroid of the cube requires that

\[
\tau_{xy} = \tau_{yx} \quad \tau_{xz} = \tau_{zx} \quad \tau_{yz} = \tau_{zy}
\]  

\[\text{...Eq. 1}\]

• Nine stress components can now reduce to six independent quantities $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$, $\tau_{xz}$, and $\tau_{zy}$, which can be written as

\[
\sigma_{ij} = \begin{pmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_y & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_z
\end{pmatrix}
\]  

\[\text{...Eq. 2}\]
**State of stress in two dimensions**

(Plane stress)

**Definition:** Plane stress is a stress condition in which the stresses are zero in one of the primary directions.

- In a **thin plate** where load will be on the plane of the plate and there will be no stress acting perpendicular to the surface of the plate.

- The stress system consists of two normal stresses $\sigma_x$ and $\sigma_y$ and a shear stress $\tau_{xy}$.
Stress on oblique plane

- Consider an **oblique plane** normal to the plane of the paper crossing \( x \) and \( y \) axis.
- The direction \( x' \) is normal to the **oblique plane** and \( y' \) direction is lying in the **oblique plane**.
- The normal stress \( \sigma \) and shear stress \( \tau \) are acting on this plane and \( A \) is the area on the **oblique plane**.
- \( S_x \) and \( S_y \) denote the \( x \) and \( y \) components of the total stress acting on the inclined face. The direction cosines between \( x' \) and the \( x \) and \( y \) axes are \( l \) and \( m \), hence \( l = \cos \theta \) and \( m = \sin \theta \).

**Summation of the forces**

\[
\begin{align*}
S_x A &= \sigma_x A_l + \tau_{xy} A_m \\
S_y A &= \sigma_y A_m + \tau_{xy} A_l \\
S_x &= \sigma_x \cos \theta + \tau_{xy} \sin \theta \\
S_y &= \sigma_y \sin \theta + \tau_{xy} \cos \theta
\end{align*}
\]
Stress on oblique plane

- The components of $S_x$ and $S_y$ in the direction of the normal stress $\sigma$ are:

$$S_{xN} = S_x \cos \theta \quad \text{and} \quad S_{yn} = S_y \sin \theta$$

- The **normal stress** acting on the oblique plane is given by:

$$\sigma_x = S_x \cos \theta + S_y \sin \theta$$

$$\sigma_x = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta$$

...Eq. 3

- The **shearing stress** on the oblique plane is given by:

$$\tau_{xy} = S_y \cos \theta - S_x \sin \theta$$

$$\tau_{xy} = \tau_{xy} (\cos^2 \theta - \sin^2 \theta) + (\sigma_y - \sigma_x) \sin \theta \cos \theta$$

...Eq. 4

- The stress $\sigma_{y'}$ can be found by substituting $\theta + \pi/2$ for $\theta$, we then have:

$$\sigma_{y'} = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2\tau_{xy} \sin \theta \cos \theta$$

...Eq. 5

If stresses in an $xy$ coordinate system and the angle $\theta$ are known, we will get the stresses in any $x'y'$ coordinate.
Stress on oblique plane

• **Equations 3-5** can be expressed in terms of **double angle** $2\theta$.

\[
\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\
\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \\
\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\]

... **Eq. 6**

... **Eq. 7**

... **Eq. 8**

**Note:** $\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y \rightarrow$ Thus **the sum of the normal stresses** on two perpendicular planes is an **invariant quantity** and independent of orientation or angle $\theta$.

**Eq. 3-8** describe the **normal stress** and **shear stress** on any plane through a point in a body subjected to a **plane-stress situation**.
Variation of normal stress and shear stress with $\theta$ for the biaxial-plane stress situation.

1) When $\tau$ is zero $\rightarrow$ give max and min values of $\sigma$.

2) The max and min values of $\sigma$ and $\tau$ occur at angles which are $90^\circ$ apart.

3) The max $\tau$ occurs at an angle halfway between the max and min $\sigma$.

4) The variation of $\sigma$ and $\tau$ occurs in the form of a sine wave, with a period of $\theta = 180^\circ$.

The relationships are valid for any state of stress.
Principal stresses

- When there is no shear stresses acting on the planes → giving the maximum normal stress acting on the planes.
- These planes are called the **principal planes**, and stresses normal to these planes are the **principal stresses** $\sigma_1$, $\sigma_2$ and $\sigma_3$ which in general do not coincide with the **cartesian-coordinate axes** $x$, $y$, $z$. Directions of principal stresses are 1, 2 and 3.

**Biaxial-plane stress condition**
- Two principal stresses, $\sigma_1$ and $\sigma_2$.

**Triaxial-plane strain condition**
- Three principal stresses, $\sigma_1$, $\sigma_2$ and $\sigma_3$, where $\sigma_1 > \sigma_2 > \sigma_3$. 

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Tapany Udomphol
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Maximum and minimum principal stresses in biaxial state of stress

• On a principal plane there is no shear stress; thereby, \( \tau_{x'y'} = 0 \)

From Eq. 5

\[
\tau_{xy} \left( \cos^2 \theta - \sin^2 \theta \right) + \left( \sigma_y - \sigma_x \right) \sin \theta \cos \theta = 0
\]

\[
\frac{\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta} = \frac{\frac{1}{2} \left( \sin 2\theta \right)}{\cos 2\theta} = \frac{1}{2} \tan 2\theta
\]

\[
\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}
\]

…Eq. 9

• Since \( \tan 2\theta = \tan(\pi + 2\theta) \), Eq. 9 has two roots, \( \theta_1 \) and \( \theta_2 = \theta_1 + n\pi/2 \). These roots define two mutually perpendicular planes which are free from shear.

• The maximum and minimum principal stresses for biaxial state of stress are given by

\[
\sigma_{\text{max}} = \sigma_1 = \sigma_x + \sigma_y \pm \left[ \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \frac{\tau_{xy}^2}{4} \right]^{1/2} \quad \text{…Eq. 10}
\]

\[
\sigma_{\text{min}} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \left[ \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \frac{\tau_{xy}^2}{4} \right]^{1/2}
\]
**Directions of the maximum principal stress**

- The largest principal stress \( \sigma_1 \) will lie between the largest *normal stress* and the *shear diagonal*.

- (If there is no shear, \( \sigma_x = \sigma_1 \), and if there is only shear the principal stress \( \sigma_1 \) would exist along the shear diagonal. If both normal and shear stresses act on the element, then \( \sigma_1 \) lies between the influence of these two effects.)
The maximum shear stress in biaxial stress

• The maximum shear stress can be found by differentiating Eq. 8 and set to zero.

\[
\frac{d\tau_{xy}}{d\theta} = (\sigma_y - \sigma_x)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0
\]

\[
\tan 2\theta_s = \frac{\sigma_y - \sigma_x}{2\tau_{xy}} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad \ldots\text{Eq. 11}
\]

**Note:** \(\tan 2\theta_s\) is the negative reciprocal of \(\tan 2\theta_n\).

This means that \(2\theta_s\) and \(2\theta_n\) are orthogonal and that \(\theta_s\) and \(\theta_n\) are separated in space by 45°.

• The magnitude of the maximum shear stress is given by

\[
\tau_{\text{max}} = \pm \left[ \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \right]^{1/2}
\]
**Example:** The state of stress is given by $\sigma_x = 25p$ and $\sigma_y = 5p$ plus shearing stresses $\tau_{xy}$. On a plane at $45^\circ$ counterclockwise to the plane on which $\sigma_x$ acts on the state of stress is 50 MPa tension and 5 MPa shear. Determine the values of $\sigma_x$, $\sigma_y$, $\tau_{xy}$.

From Eq.6

$$\sigma_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$50 \times 10^6 = \frac{25p + 5p}{2} + \frac{25p - 5p}{2} \cos 90^\circ + \tau_{xy} \sin 90^\circ$$

$$50 \times 10^6 = 15p + \tau_{xy}$$

From Eq.8

$$\tau_{xy} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$5 \times 10^6 = \left(\frac{5p - 25p}{2}\right) \sin 90^\circ + \tau_{xy} \cos 90^\circ$$

$$p = -5 \times 10^6 \text{ Pa}$$

$$\sigma_x = 25(-5 \times 10^6) = -12.5 \text{ MPa}$$

$$\sigma_y = 5(p) = 2.5 \text{ MPa}$$

$$\tau_{xy} = 50 \times 10^6 - 15(-5 \times 10^5)$$

$$\tau_{xy} = 57.5 \text{ MPa}$$

Since $\sigma_x + \sigma_y = \sigma_x' + \sigma_y'$

$$\sigma_y' = \sigma_x + \sigma_y - \sigma_x'$$

$$\sigma_y' = -12.5 - 2.5 - 50 = -65 \text{ MPa}$$
Mohr’s circle of stress - two dimensions

O. Mohr used a graphical method to represent the state of stress at a point on an oblique plane through the point.

From Eq. 6 and Eq. 8

\[
\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad \ldots\text{Eq. 6}
\]

\[
\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \quad \ldots\text{Eq. 8}
\]

By squaring each of these equations and adding, we have

\[
\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \quad \ldots\text{Eq. 12}
\]

Eq. 12 is in the equation of a circle of the form \((x-h)^2 + y^2 = r^2\). Therefore Mohr’s circle is a circle in \(\sigma_{x'}, \tau_{x'y'}\) coordinates which \(r = \tau_{\text{max}}\) and the centre is displaced \((\sigma_x + \sigma_y)/2\) to the right of the origin.
Mohr’s circle

Conventions

- A shear stress causing a clockwise rotation about any point in the physical element is plotted above the horizontal axis of the Mohr’s circle.
- A point on Mohr’s circle gives the magnitude and direction of the normal and shear stresses on any plane in the physical element.

- Normal stresses are plotted along the x axis, shear stresses along the y axis.
- The stresses on the planes normal to the x and y axes are plotted as points A and B.
- The shear stress is zero at points D and E, representing the values of the principal stresses $\sigma_1$ and $\sigma_2$ respectively.
- The Angle between $\sigma_x$ and $\sigma_1$ on Mohr’s circle is $2\theta$. 

Mohr’s circle for two-dimensional state of stress
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State of stress in three dimensions

- In general, a three-dimensional state of stress consists of three unequal principal stresses acting at a point, which is called a triaxial state of stress.
- If two of the three principal stresses are equal, it is cylindrical.
- If \( \sigma_1 = \sigma_2 = \sigma_3 \), it is hydrostatic or spherical.
Stress in three dimensions

• Considering an elemental free body with diagonal plane \(JKL\) of area \(A\), which is assumed to be a principal plane cutting through the unit cube.

• The principal stress \(\sigma\) is acting normal to the plane \(JKL\). The direction cosines of \(\sigma\) and \(x, y,\) and \(z\) axes is \(l, m\) and \(n\) respectively.

In equilibrium, the forces acting on each of its face must balance. \(S_x, S_y\) and \(S_z\) are the components of \(\sigma\) along the axes.

\[
\begin{align*}
S_x &= \sigma l \\
S_y &= \sigma m \\
S_z &= \sigma n \\
\end{align*}
\]

Taking summation of the forces in the \(x\) direction results in

\[
\sigma A_l - \sigma_x A l - \tau_{yx} A m - \tau_{zx} A n = 0
\]

Which reduces to

\[
(\sigma - \sigma_x) l - \tau_{yx} m - \tau_{zx} n = 0
\]

...Eq. 13
Summation of the forces in the other two directions results in

\[-\tau_{xy}l + (\sigma - \sigma_y)m - \tau_{zy}n = 0 \quad \ldots\text{Eq. 13(b)}\]
\[-\tau_{xz}l - \tau_{yz}m + (\sigma - \sigma_z)n = 0 \quad \ldots\text{Eq. 13(c)}\]

By setting the determinant of the coefficients of \(l\), \(m\) and \(n = 0\)

Will give the solution of the determinant which results in a cubic equation in \(\sigma\)

And invariant coefficients

\[
\sigma^3 - \left(\sigma_x + \sigma_y + \sigma_z\right)\sigma^2 + \left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2\right)\sigma

- \left(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2\right) = 0
\]

\[
I_1 = \sigma_x + \sigma_y + \sigma_z

I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2

I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2
\]

\[
\sigma_x + \sigma_y + \sigma_z = \sigma_x' + \sigma_y' + \sigma_z' = \sigma_1 + \sigma_2 + \sigma_3
\]
Determination of the normal stress on any oblique plane

• On any oblique plane whose normal has the direction cosines \( l, m, n \) with the \( x, y, z \) axes, The total stress on the plane \( S \) will not now be coaxial with the normal stress, and that \( S^2 = \sigma^2 + \tau^2 \).

• The total stress can be resolved into components \( S_x, S_y, S_z \), so that

\[
S^2 = S_x^2 + S_y^2 + S_z^2 \quad \ldots \text{Eq. 15}
\]

• Taking the summation of the forces in the \( x, y, z \) directions, the expressions for the orthogonal components of the total stress are given by;

\[
\begin{align*}
S_x &= \sigma_x l + \tau_{yx} m + \tau_{zx} n \\
S_y &= \tau_{xy} l + \sigma_y m + \tau_{zy} n \quad \ldots \text{Eq. 16} \\
S_z &= \tau_{xz} l + \tau_{yz} m + \sigma_z n
\end{align*}
\]

• The normal stress \( \sigma \) on the oblique plane;

\[
\sigma = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl \quad \ldots \text{Eq. 17}
\]

Substituting Eq and simplifying \( \tau_{xy} = \tau_{yx} \) Etc.
The maximum or principal shear stress

• Since plastic flow involves shearing stresses, it is important to identify the planes on which the maximum or principal shear stresses occur.

• The principal shear planes can be defined in terms of the three principal axes 1, 2, 3.

\[ \tau^2 = (\sigma_1 - \sigma_2)^2 l^2 m^2 + (\sigma_1 - \sigma_3)^2 l^2 n^2 + (\sigma_2 - \sigma_3)^2 m^2 n^2 \]

...Eq. 14

Where \( l, m \) and \( n \) are the direction cosines between the normal to the oblique plane and the principal axes.

The principal shear stresses occur for the following combination of direction cosines that bisect the angle between two of the three principal axes:

\[
\begin{array}{ccc}
0 & \pm \sqrt{\frac{1}{2}} & \pm \sqrt{\frac{1}{2}} & \tau_1 = \frac{\sigma_2 - \sigma_3}{2} \\
\pm \sqrt{\frac{1}{2}} & 0 & \pm \sqrt{\frac{1}{2}} & \tau_2 = \frac{\sigma_1 - \sigma_3}{2} \\
\pm \sqrt{\frac{1}{2}} & \pm \sqrt{\frac{1}{2}} & 0 & \tau_3 = \frac{\sigma_1 - \sigma_2}{2}
\end{array}
\]
The maximum or principal shear stress

- **Principal shear stresses** for a cube whose faces are the principal planes.

- For each pair of **principal stresses**, there are two planes of **principal shear stress**, which bisect the directions of the principal stresses.

According to convention $\sigma_1$ is the greatest principal normal stress and $\sigma_3$ is the smallest principal stress, $\tau_2$ therefore has the largest value

- The **maximum principal shear stress** $\tau_{\text{max}}$ is given by

$$\tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} \quad \text{...Eq. 18}$$
Stress tensor is used to simplify the equations for the transformation of the stress components from one set of coordinate axes to another coordinate system.

• First, we consider the transformation of a vector (first-rank tensor) from one coordinate system to another.

\[ S = S_1 i_1 + S_2 i_2 + S_3 i_3 \]

when the unit vectors \( i_1, i_2, i_3 \) are in the direction \( x_1, x_2, x_3 \).

Where \( S_1, S_2, S_3 \) are the components of \( S \) referred to the axes \( x_1, x_2, x_3 \).

• The components of \( S \) referred to the \( x'_1, x'_2, x'_3 \) is obtained by resolving \( S_1, S_2, S_3 \) along the new direction \( x'_1 \).

\[ S'_1 = S_1 \cos(x_1x'_1) + S_2 \cos(x_2x'_1) + S_3 \cos(x_3x'_1) \]

or

\[ S'_1 = a_{11}S_1 + a_{12}S_2 + a_{13}S_3 \]

Where \( a_{11} \) is the direction cosine between \( x'_1 \) and \( x_1 \), \( a_{12} \) is the direction cosine between \( x'_1 \) and \( x_2 \), etc..

\[ S'_2 = a_{21}S_1 + a_{22}S_2 + a_{23}S_3 \]

\[ S'_3 = a_{31}S_1 + a_{32}S_2 + a_{33}S_3 \]

...Eq. 19
• We could write the equations (from last slide) as

\[
S'_1 = \sum_{j=1}^{3} a_{1j} S_j \quad S'_2 = \sum_{j=1}^{3} a_{2j} S_j \quad S'_3 = \sum_{j=1}^{3} a_{3j} S_j
\]

• These three equations could be combined by writing;

\[
S'_i = \sum_{j=1}^{3} a_{ij} S_j (i = 1, 2, 3) = a_{i1} S_1 + a_{i2} S_2 + a_{i3} S_3 \quad \ldots \text{Eq. 20}
\]

• In greater brevity, the equation is obtained by writing in the Einstein suffix notation.

\[
S'_i = a_{ij} S_j \quad \ldots \text{Eq. 21}
\]

• The suffix notation \((j)\) indicates \textbf{summation} when a suffix occurs twice in the same term. The summation will take place over \(j\).

• The transformation of the stress tensor \(\sigma_{ij}\) from the \(x_1, x_2, x_3\) system of axes to the \(x'_1, x'_2, x'_3\) axes is given by

\[
\sigma_{kl} = a_{ki} a_{lj} \sigma_{ij}
\]
Rank of tensors

- **Scalars** are tensors of rank zero, in which their quantity remains unchanged with the transformation of axes.

- **Vectors** are tensors of first rank.

- **Physical quantities** such as stress, strain and many other quantities are second-rank tensors which transform with coordinate axes.

The number of components required to specify a quantity is $3^n$, when $n$ is the rank of the tensor.
Mohr’s circle – three dimensions

*Mohr’s circle in three dimensions* show how a triaxial state of stress is presented.

- All the possible stress conditions within the body fall within the *shaded area* between the circles.
- The $y$ axis is the *shear stress* $\tau$ and the $x$ axis is the *normal stress* $\sigma$.
- $\sigma_1$, $\sigma_2$, $\sigma_3$ are shown on the $x$ axis and $\tau_1$, $\tau_2$, $\tau_3$ are shown on the $y$ axis.
Mohr’s circle for various states of stress

- **Mohr’s circle** give a geometrical representation of the equations that express the transformation of stress components to different sets of axes.
- The introduction of $\sigma_2$ at a right angle to $\sigma_1$ results in a reduction in the principal shear stress, but $\tau_2$ remains similar see *fig (c).*
- The *maximum shear stress* is reduced appreciably when the *third principal stress* is introduced, see *fig (d).*
- Hydrostatic tension $\sigma_1 = \sigma_2 = \sigma_3$, Mohr’s circle reduces to a point with no shear stress.
Biaxial and triaxial tension stresses effectively reduce the shear stresses and this results in a **considerable decrease in ductility** of the material, because plastic deformation is produced by shear stresses.

Uniaxial tension plus biaxial compression stresses produce high value of shear stress and contribute to an excellent opportunity to deform plastically without fracturing.

**Ex:** forming by metal extrusion gives better ductility than simple uniaxial tension.
Description of strain at a point

• Deformation of a solid may be made up of dilatation (change in volume), or distortion (change in shape). This results in displacement of points in a continuum body.

• Consider a solid body in fixed coordinates $x, y, z$ with a displacement from point $Q$ to $Q'$.  
• The components of displacement are $u, v, w$.  
• The displacement of $Q$ is the vector $u_Q = f(u, v, w)$.

• Displacement is a function of distance, $u_i = f(x_i)$ and for elastic solid and small displacement, $u_i$ is a linear function of $x_i$.

Note: in other materials the displacement may not be linear with distance, which leads to cumbersome mathematical relationships.
One-dimensional strain

- Consider a simple one-dimensional strain, which has been deformed from the original distance $AB$ to $A'B'$.
- Displacement $u$ is in one dimension (as a function of $x$).
- The normal strain is given by

$$ e_x = \frac{\Delta L}{L} = \frac{A'B' - AB}{AB} = \frac{dx + \frac{\partial u}{\partial x} dx - dx}{dx} = \frac{\partial u}{\partial x} $$

...Eq. 22

The displacement in one dimensional case is given by

$$ u = e_x x $$

...Eq. 23
Three-dimensional strain

• In three dimensional strain, each of the component of the displacement will be linearly related to each of the three initial coordinates of the point.

• Three coefficients for the normal strains,

\[
e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{zz} = \frac{\partial w}{\partial z}
\]

• Consider an angular distortion of an element in the \(xy\) plane by shearing stresses, we have

\[
e_{xy} = \frac{DD'}{DA} = \frac{\partial u}{\partial y}, \quad e_{yx} = \frac{BB'}{AB} = \frac{\partial v}{\partial x}
\]

\[
e_{ij} = \begin{vmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}
\]

\[
u = e_{yx} x + e_{yy} y + e_{yz} z
\]
\[
w = e_{zx} x + e_{zy} y + e_{zz} z
\]

\[
u = e_{yx} x + e_{yy} y + e_{yz} z
\]

or \( u_i = e_{ij} x_j \)

...Eq. 24

...Eq. 25

Angular distortion of an element
Strain tensor and rotation tensor

• In general, displacement components such as $e_{xy}$, $e_{yx}$... produce both **shear strain** and **rigid-body rotation**.

• From tensor theory, any second-rank tensor can be decomposed into a symmetric tensor and an anisymmetric tensor.

\[
e_{ij} = \frac{1}{2} (e_{ij} + e_{ji}) + \frac{1}{2} (e_{ij} - e_{ji})
\]

\[
e_{ij} = \varepsilon_{ij} + \omega_{ij}
\]

...Eq. 26

where

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

\[
\omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)
\]

(a) Pure shear without rotation.
(b) Pure rotation without shear.
(c) Simple shear.

Strain tensor

The general displacement equations

\[
u_i = \varepsilon_{ij} x_j + \omega_{ij} x_j
\]

...Eq. 27
Shear strain

The shear strain \( \gamma \) was defined as the total angular change from a right angle.

\[
\gamma = e_{xy} + e_{yx} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}
\]

And the definition of shear strain, \( \gamma_{ij} = \varepsilon_{ij} \), is called the engineering shear strain.

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\]
\[
\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
\]
\[
\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}
\]

Since strain is a second-rank tensor, following the transformation of stress previously done, the strain tensor may be transformed one set of coordinate axes of coordinate axes to a new system of axes by

\[
\varepsilon_{kl} = a_{ki} a_{lj} \varepsilon_{ij}
\]
Principal shear strain

Following Eq. 17, substituting \( \varepsilon \) for \( \sigma \) and \( \sqrt{2} \) for \( \tau \). The normal strain on an oblique plane is given by

\[
\varepsilon = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{xy} l m + \gamma_{yz} m n + \gamma_{xz} n l \\
\]

…Eq. 31

Similar to stress, the direction of the principal strains coincide with the principal stress directions. The three principal strains are the roots of the cubic equation.

\[
\varepsilon^3 - I_1 \varepsilon^2 + I_2 \varepsilon - I_3 = 0 \\
\]

…Eq. 32 Where

\[
\begin{align*}
I_1 &= \varepsilon_x + \varepsilon_y + \varepsilon_z \\
I_2 &= \varepsilon_x \varepsilon_y + \varepsilon_y \varepsilon_z + \varepsilon_z \varepsilon_x - \frac{1}{4} \left( \gamma_{xy}^2 + \gamma_{zx}^2 + \gamma_{yz}^2 \right) \\
I_3 &= \varepsilon_x \varepsilon_y \varepsilon_z + \frac{1}{4} \left( \varepsilon_x \gamma_{yz}^2 + \varepsilon_y \gamma_{zx}^2 + \varepsilon_z \gamma_{xy}^2 \right) \\
\end{align*}
\]

…Eq. 33

The maximum shearing strains can be obtained from

\[
\begin{align*}
\gamma_1 &= \varepsilon_2 - \varepsilon_3 \\
\gamma_{\text{max}} &= \gamma_2 = \varepsilon_1 - \varepsilon_3 \\
\gamma_3 &= \varepsilon_1 - \varepsilon_2 \\
\end{align*}
\]

…Eq. 34
Strain tensor can be divided into a hydrostatic or mean strain and a strain deviator.

1) Hydrostatic or mean strain (volume change)

\[ \varepsilon_m = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3} = \frac{\varepsilon_{kk}}{3} = \frac{\Delta}{3} \]

Where \( \Delta \) is the volume strain (change in volume).

2) Strain deviator (shape change)

We can do by subtracting \( \varepsilon_m \) from the normal strain components, thus

\[ \varepsilon_{ij} = \begin{bmatrix} \varepsilon_x - \varepsilon_m & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y - \varepsilon_m & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z - \varepsilon_m \end{bmatrix} \]

These strains represent elongations or contractions along the principal axes that change the shape of the body at constant volume.
Strain measurement

- Strain can be measured by using a bonded-wire resistance gauge or strain gauge.
- When the body is deformed, the wires in the strain gauge are strained and their electrical resistance is altered.
- The change in resistance, which is proportional to strain can therefore be determined.
- Strain gauges can make only direct readings of linear strain, while shear strains must be determined indirectly.

Typical strain gauge rosettes.

- Rectangular
- Delta
Hydrostatic and deviator components of stress

Similar to strain tensor, the total stress tensor can be divided into

1) **Hydrostatic or mean stress tensor**, $\sigma_m$, which involves only pure tension or compression. $\rightarrow$ Produce *elastic volume changes*.

$$\sigma_m = \frac{\sigma_{kk}}{3} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \quad \text{...Eq. 37}$$

2) **Deviator stress tensor** $\sigma'_{ij}$. Which represents the shear stress in the total state of stress. $\rightarrow$ important in causing *plastic deformation*.
Deviator of stress tensor

Since the decomposition of the stress tensor is given by

\[ \sigma_{ij} = \sigma_{ij}' + \frac{1}{3} \delta_{ij} \sigma_{kk} \]  \hspace{1cm} \text{...Eq. 38}

**Stress deviator** involves shear stress. For example, referring \( \sigma_{ij}' \) to a system of principal axes.

\[
\begin{align*}
\sigma_1' &= \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3} = \frac{(\sigma_1 - \sigma_2) + (\sigma_1 - \sigma_3)}{3} \\
\sigma_1' &= \frac{2}{3} \left( \frac{\sigma_1 - \sigma_2}{2} + \frac{\sigma_1 - \sigma_3}{2} \right) = \frac{2}{3} (\tau_3 + \tau_2) \\
\end{align*}
\]  \hspace{1cm} \text{...Eq. 39}

Where \( \tau_3 \) and \( \tau_2 \) are principal shearing stresses.
Deviator of stress tensor

- The principal values of the stress deviator are the roots of the cubic equation:

\[
(\sigma')^3 - J_1 (\sigma')^2 - J_2 \sigma' - J_3 = 0
\]

Where \( J_1, J_2, J_3 \) are the invariants of the deviator stress tensor.

\[
J_1 = (\sigma_x - \sigma_m) + (\sigma_y - \sigma_m) + (\sigma_z - \sigma_m) = 0
\]

\[
J_2 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2 - \sigma_x' \sigma_y' - \sigma_y' \sigma_z' - \sigma_z' \sigma_x'
\]

\[
J_2 = \frac{1}{6} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2) \right]
\]
Elastic stress – strain relations

The **elastic stress** is linearly related to **elastic strain** following **Hooke’s law**.

Where $E$ is the modulus of elasticity in tension or compression.

\[ \sigma_x = E \varepsilon_x \]  

...Eq. 40

However, during linear extension, i.e., in $x$ axis, the contraction in the transverse $y$ and $z$ direction causes a constant fraction of the strain in the longitudinal direction known as **Poisson’s ratio** $\nu$.

\[ \varepsilon_y = \varepsilon_z = -\nu \varepsilon_x = -\frac{\nu \sigma_x}{E} \]  

...Eq. 41

**Note:** for most metals $\nu \sim 0.33$
When a material is deformed by an external loading, work done during elastic deformation is stored as elastic energy and will be recovered when the load is released. → Strain energy.

\[ U = \frac{1}{2} P \delta \]  

...Eq. 42

Where \( P/2 \) is the average force from zero. \( \delta \) is the extension.

For linear elastic (i.e., \( x \) axis) then Hooke's law is applied (\( \sigma = E \varepsilon \))

\[ U_o = \frac{1}{2} \sigma_x \varepsilon_x = \frac{1}{2} \frac{\sigma_x^2}{E} = \frac{1}{2} \varepsilon_x^2 E \]  

...Eq. 43
Stress concentration

- Discontinuity such as a **hole** or a **notch** results in **non-uniform stress distribution** at the vicinity of the discontinuity. **Stress concentration or stress raiser.**

- The distribution of the axial stress reaches a high value at the edges of the hole and drops rapidly with distance away from the hole.

- The stress concentration is expressed by a theoretical stress-concentration factor \( K_t \).

\[
K_t = \frac{\sigma_{\text{max}}}{\sigma_{\text{nom}}} \quad \text{...Eq. 44}
\]

**Stress distributions due to (a) circular hole and (b) elliptical hole.**
Stress concentration at a circular hole in a plate

• For the circular hole in a plate subjected to an axial load, a radial stress is produced as well as a longitudinal stress.

• From elastic analysis, the stresses can be expressed as:

\[
\sigma_r = \frac{\sigma}{2} \left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma}{2} \left(1 + 3 \frac{a^4}{r^4} - 4 \frac{a^2}{r^2}\right) \cos 2\theta
\]

\[
\sigma_\theta = \frac{\sigma}{2} \left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma}{2} \left(1 + 3 \frac{a^4}{r^4}\right) \cos 2\theta
\]

\[
\tau = -\frac{\sigma}{2} \left(1 - 3 \frac{a^4}{r^4} + 2 \frac{a^2}{r^2}\right) \sin 2\theta
\]

The maximum stress occurs at point A, when \(\theta = \frac{\pi}{2}\) and \(r = a\)

\[
\sigma_\theta = 3\sigma = \sigma_{max}
\]

The theoretical stress-concentration factor = 3
Stress concentration at an elliptical hole in a plate

In the case of an elliptical hole in a plate, the maximum stress at the ends of the hole is given by the equation

\[
\sigma_{\text{max}} = \sigma \left(1 + 2 \frac{a}{b}\right) \quad \text{...Eq. 47}
\]

Therefore, a very sharp crack normal to the tensile direction will result in a very high stress concentration.
Stress concentrations for different geometrical shapes
Stress raiser in brittle and ductile materials

- The effect of stress raiser is much more pronounced in brittle materials than in ductile materials.

**Ductile materials**
- Plastic deformation occurs when the yield stress is exceeded at the point of maximum stress.
- In ductile materials, further increase in load produce strain hardening (work hardening) → redistribution of stress → the materials will not develop the full theoretically stress-concentration factor.

**Brittle materials**
- Stress redistribution will not occur in to any extent in brittle materials. → stress concentration of close to the theoretical value will result.
Finite element method

- **Finite element method (FEM)** is a very powerful technique for determining stresses and deflections in structures too complex to analyse by strictly analytical methods.

- The structure is divided into a network of small elements connected to each other at node points. One node has one degree of freedom (see fig showing two and three dimensional element)

![Diagram of finite element method](https://via.placeholder.com/150)

(a) Simple rectangular element

(b) Two elements joined to model a structure

Some common elements used in FEM analysis
Finite element method

• A finite element solution involves *calculating* the stiffness matrices for every element in the structure.

• A cumbersome part of the finite element solution is the **preparation of the input data**. Required data such as topology of the element mesh, node numbers, coordinates of the node points.

*Propeller designed by means of FEM analysis*
References

• www.ndsu.nodak.edu
• www.kockums.se
• www-indentec.com
• www.enduratec.com
• www.bactechnologies.com
• www.twi.co.uk
• www.jaeri.go.jp
• www.hghouston.com
• www.ship-technology.com