2.4 Base Excitation

- Important class of vibration analysis
  - Preventing excitations from passing from a vibrating base through its mount into a structure
- Vibration isolation
  - Vibrations in your car
  - Satellite operation
  - Disk drives, etc.

FBD of SDOF Base Excitation

\[
\sum F = -k(x-y) - c(\dot{x} - \dot{y}) = m\ddot{x}
\]

SDOF Base Excitation (cont)

Assume: \( y(t) = Y \sin(\Omega t) \) and plug into Equation (2.61)

\[
m\dddot{x} + c\ddot{x} + kx = c\Omega^2 Y \cos(\Omega t) + kY \sin(\Omega t)
\]

For a car, \( \omega = \frac{2\pi}{\tau} = \frac{2\pi V}{\lambda} \)

The steady-state solution is just the superposition of the two individual particular solutions (system is linear).

\[
\dot{x} + 2\zeta_0 \omega_0 \dot{x} + \omega_0^2 x = \frac{f_0}{\omega_0^2 - \omega^2} \sin(\omega t)
\]

Particular Solution (sine term)

With a sine for the forcing function, use rectangular form to make it easier to add the cos term.

\[
x_{ps} = A_s \cos(\omega t) + B_s \sin(\omega t) = X_s \sin(\omega t - \phi_s)
\]

where

\[
A_s = \frac{-2\zeta_0 \omega_0 f_{0s}}{(\omega_0^2 - \omega^2)^2 + (2\zeta_0 \omega_0 \omega)^2}
\]

\[
B_s = \frac{(\omega_0^2 - \omega^2) f_{0s}}{(\omega_0^2 - \omega^2)^2 + (2\zeta_0 \omega_0 \omega)^2}
\]
Particular Solution (cos term)

With a cosine for the forcing function, we showed

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = f_{oc} \cos \omega t \]

\[ x_{pc} = A_c \cos \omega t + B_c \sin \omega t = X_c \cos(\omega t - \phi_c) \]

where

\[ A_c = \frac{(\omega_n^2 - \omega^2) f_{oc}}{\left(\omega_n^2 - \omega^2\right) + (2\zeta \omega_n \omega)^2} \]

\[ B_c = \frac{2\zeta \omega_n \omega f_{oc}}{\left(\omega_n^2 - \omega^2\right) + (2\zeta \omega_n \omega)^2} \]

Magnitude X/Y

Now add the sin and cos terms to get the magnitude of the full particular solution

\[ X = \sqrt{f_{oc}^2 + f_{os}^2} = \omega_n Y \sqrt{\frac{(2\zeta \omega)^2 + \omega_n^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2}} \]

where \( f_{oc} = 2\zeta \omega_n \omega Y \) and \( f_{os} = \omega_n^2 Y \)

if we define \( r = \frac{\omega}{\omega_n} \) this becomes

\[ X = Y \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \]

\[ \frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \]

From the plot of relative Displacement Transmissibility observe that:

- X/Y is called Displacement Transmissibility Ratio
- Potentially severe amplification at resonance
- Attenuation for \( r > \sqrt{2} \) Isolation Zone
- If \( r < \sqrt{2} \) transmissibility decreases with damping ratio Amplification Zone
- If \( r >> 1 \) then transmissibility increases with damping ratio \( X_p \sim 2Y \zeta / r \)
Next examine the *Force Transmitted to the mass* as a function of the frequency ratio

\[ F_r = -k(x - y) - c(\dot{x} - \dot{y}) = m\ddot{x} \]

From FBD

At steady state, \( x(t) = X \cos(\omega t - \phi) \),

so \( \ddot{x} = -\omega^2 X \cos(\omega t - \phi) \)

\[ |F_r| = m\omega^2 X = kr^2 X \]

**Figure 2.14**

Plot of Force Transmissibility (in dB) versus frequency ratio

**Figure 2.15** Comparison between force and displacement transmissibility

**Example 2.4.1:** Effect of speed on the amplitude of car vibration
Model the road as a sinusoidal input to base motion of the car model

Approximation of road surface:
\[ y(t) = (0.01 \text{ m}) \sin \omega_b t \]

\[ \omega_b = v(\text{km/hr}) \left( \frac{1}{0.006 \text{ km}} \right) \left( \frac{3600 \text{ s}}{\text{cycle}} \right) \left( \frac{2 \pi \text{ rad}}{\text{cycle}} \right) = 0.2909 \nu \text{ rad/s} \]

\[ \omega_b(20\text{km/hr}) = 5.818 \text{ rad/s} \]

From the data given, determine the frequency and damping ratio of the car suspension:

\[ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \times 10^4 \text{ N/m}}{1007 \text{ kg}}} = 6.303 \text{ rad/s} \quad (\approx 1 \text{ Hz}) \]

\[ \zeta = \frac{c}{2\sqrt{km}} = \frac{2000 \text{ Ns/m}}{2\sqrt{(4 \times 10^4 \text{ N/m})(1007 \text{ kg})}} = 0.158 \]

What happens as the car goes faster? See Table 2.1.

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Example 2.4.2: Compute the force transmitted to a machine through base motion at resonance

From (2.77) at \( r = 1 \):

\[ F_r = \frac{kY}{2\zeta \sqrt{1 + 4\zeta^2}} \]

From given \( m, c, \) and \( k \):

\[ \zeta = \frac{c}{2\sqrt{km}} = \frac{900}{2\sqrt{40,000 \times 3000}} \approx 0.04 \]

From measured excitation \( Y = 0.001 \text{ m} \):

\[ F_r = \frac{kY}{2\zeta \sqrt{1 + 4\zeta^2}} = \frac{(40,000 \text{ N/m})(0.001 \text{ m})}{2(0.04)} \sqrt{1 + 4(0.04)^2} = 501.6 \text{ N} \]

- Gyros
- Cryo-coolers
- Tires
- Washing machines

\( e = \) eccentricity
\( m_o = \) mass unbalance
\( \omega_r = \) rotation frequency
Rotating Unbalance (cont)

What force is imparted on the structure? Note it rotates with x component:

\[ x_r = e \sin \omega_r t \]

\[ \Rightarrow a_x = \ddot{x}_r = -m_0 e \omega_r^2 \sin \omega_r t \]

From sophomore dynamics, 

\[ R_x = m_0 a_x = -m_0 e \omega_r^2 \sin \theta = -m_0 e \omega_r^2 \sin \omega_r t \]
\[ R_y = m_0 a_y = -m_0 e \omega_r^2 \cos \theta = -m_0 e \omega_r^2 \cos \omega_r t \]

The problem is now just like any other SDOF system with a harmonic excitation

\[ m \dddot{x} + c \ddot{x} + kx = m_0 e \omega_r^2 \sin \omega_r t \]  (2.82)

or  
\[ \dot{x} + 2 \zeta \omega_n \dot{x} + \omega_n^2 x = \frac{m_0}{m} e \omega_r^2 \sin \omega_r t \]

Note the influences on the forcing function (we are assuming that the mass m is held in place in the y direction as indicated in Figure 2.18)

Rotating Unbalance (cont)

• Just another SDOF oscillator with a harmonic forcing function
• Expressed in terms of frequency ratio \( r \)

\[ x_p(t) = X \sin(\omega_r t - \phi) \]  (2.83)

\[ X = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1 - r^2)^2 + (2 \zeta r)^2}} \]  (2.84)

\[ \phi = \tan^{-1} \left( \frac{2 \zeta r}{1 - r^2} \right) \]  (2.85)

Figure 2.20: Displacement magnitude vs frequency caused by rotating unbalance
Example 2.5.1: Given the deflection at resonance (0.1m), $\zeta = 0.05$ and a 10% out of balance, compute $e$ and the amount of added mass needed to reduce the maximum amplitude to 0.01 m.

At resonance $r = 1$ and

$$\frac{mX}{m_0e} = \frac{1}{2\zeta} = \frac{1}{2(0.05)} \Rightarrow \frac{10 \times 0.1 \text{ m}}{e} = \frac{1}{2\zeta} = 10 \Rightarrow e = 0.1 \text{ m}$$

Now to compute the added mass, again at resonance;

$$\frac{m + \Delta m}{m_0} \left(\frac{X}{0.1 \text{ m}}\right) = 10 \quad \text{Use this to find } \Delta m \text{ so that } X \text{ is 0.01:}$$

$$\frac{m + \Delta m}{m_0} \left(\frac{0.01 \text{ m}}{0.1 \text{ m}}\right) = 10 \Rightarrow \frac{m + \Delta m}{(0.1)m} = 100 \Rightarrow \Delta m = 9 \text{ m}$$

Here $m_0$ is 10% or 0.1 m.

Example 2.5.2 Helicopter rotor unbalance

Given

$$k = 1 \times 10^5 \text{ N/m}$$

$$m_{\text{tail}} = 60 \text{ kg}$$

$$m_{\text{rot}} = 20 \text{ kg}$$

$$m_0 = 0.5 \text{ kg}$$

$$\zeta = 0.01$$

Compute the deflection at 1500 rpm and find the rotor speed at which the deflection is maximum

Example 2.5.2 Solution

The rotating mass is 20 + 0.5 or 20.5. The stiffness is provided by the Tail section and the corresponding mass is that determined in Example 1.4.4. So the system natural frequency is

$$\omega_n = \sqrt{\frac{k}{m + m_{\text{tail}}}} = \sqrt{\frac{10^5 \text{ N/m}}{20.5 + \frac{60 \text{ kg}}{3}}} = 46.69 \text{ rad/s}$$

The frequency of rotation is

$$\omega_r = 1500 \text{ rpm} = 1500 \text{ rev/min} \frac{2\pi \text{ rad}}{\text{min} 60 \text{ s} \text{ rev}} = 157 \text{ rad/s}$$

$$\Rightarrow r = \frac{157 \text{ rad/s}}{49.49 \text{ rad/s}} = 3.16$$

Now compute the deflection at $r = 3.16$ and $\zeta = 0.01$ using eq (2.84)

$$X = \frac{m_0e}{m} \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$= \frac{(0.5 \text{ kg})(0.15 \text{ m})}{20.5 \text{ kg}} \frac{(3.16)^2}{\sqrt{(1 - (3.16)^2)^2 - (2(0.01)(3.16))^2}} = 0.004 \text{ m}$$

At around $r = 1$, the max deflection occurs:

$$r = 1 \Rightarrow \omega_r = 49.69 \text{ rad/s} = 49.69 \frac{\text{rad}}{s} \frac{\text{rev}}{2\pi \text{ rad} \text{ min}} = 474.5 \text{ rpm}$$

$$\Rightarrow X = \frac{(0.5 \text{ kg})(0.15 \text{ m})}{20.5 \text{ kg}} \frac{1}{2(0.01)} = 0.183 \text{ m or 18.3 cm}$$
2.6 Measurement Devices

- A basic transducer used in vibration measurement is the accelerometer.
- This device can be modeled using the base equations developed in the previous section.

\[ F = -k(x-y) - c(\dot{x} - \dot{y}) = m\ddot{x} \]

\[ m\ddot{x} = -c(\dot{x} - \dot{y}) - k(x - y) \]

(2.86) and (2.61)

Here, \( y(t) \) is the measured response of the structure.

Magnitude and sensitivity plots for accelerometers.

Base motion applied to measurement devices

Let \( z(t) = y(t) - y(t) \) (2.87):

\[ m\ddot{z} + c\dot{z} + kz(t) = m\Omega^2 Y \cos \Omega t \] (2.88)

\[ Y = \frac{Z}{\sqrt{1 - r^2} + (2\zeta r)^2} \]

(2.90)

These equations should be familiar from base motion. Here they describe measurement!

Effect of damping on proportionality constant

Fig 2.27

In the accel region, output voltage is nearly proportional to displacement.

Home Work-Chapter 2

2.6, 2.7, 2.9
2.11, 2.15, 2.19
2.24, 2.28
2.30, 2.35
2.40, 2.44
2.51, 2.52, 2.53, 2.57, 2.58