Chapter 6

Control Systems Design by Root-Locus Method

Lag-Lead Compensation

Lag—lead Compensation Techniques Based on the Root-Locus Approach.

$$G_c(s) = K_c\left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right), \qquad (\gamma > 1, \beta > 1)$$

In designing lag—lead compensators, we consider two cases where $\gamma \neq \beta$ and $\gamma = \beta$

Case I $\gamma \neq \beta$. In this case, the design process is a combination of the design of the lead compensator and that of the lag compensator. The design procedure for the lag—lead compensator follows:

2

Lag—lead Compensation Techniques Based on the Root-Locus Approach.

1. From the given performance specifications, determine the desired location for the dominant closed-loop poles.
2. Using the uncompensated open-loop transfer function $G(s)$, determine the angle deficiency ϕ if the dominant closed-loop poles are to be at the desired location. The phase-lead portion of the lag—lead compensator must contribute this angle ϕ.
3. Assuming that we later choose T_2 sufficiently large so that the magnitude of the lag portion

$$\left|\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right|$$

3

Lag—lead Compensation Techniques Based on the Root-Locus Approach.

is approximately unity, where $s = s_1$ is one of the dominant closed-loop poles, choose the values of T_1 and γ from the requirement that

$$\angle \frac{s_1+\frac{1}{T_1}}{s_1+\frac{\gamma}{T_1}} = \phi$$

The choice of T_1 and γ is not unique. (Infinitely many sets of T_1 and γ are possible.) Then determine the value of K_c from the magnitude condition:

$$\left|K_c \frac{s_1+\frac{1}{T_1}}{s_1+\frac{\gamma}{T_1}} G(s_1)\right| = 1$$

4

Lag—lead Compensation Techniques Based on the Root-Locus Approach.

4. If the static velocity error constant K_v is specified, determine the value of β to satisfy the requirement for K_v. The static velocity error constant K_v is given by

$$K_v = \lim_{s\to 0} sG_c(s)G(s)$$

$$= \lim_{s\to 0} sK_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right)G(s)$$

$$= \lim_{s\to 0} sK_c \frac{\beta}{\gamma} G(s)$$

where K_c and γ are already determined in step 3. Hence, given the value of K_v the value of β can be determined from this last equation. Then, using the value of β thus determined,

5

Lag—lead Compensation Techniques Based on the Root-Locus Approach.

Choose the value of T_2 such that

$$\left|\frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}}\right| \doteq 1$$

$$-5° < \angle \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} < 0°$$

6

Lag—lead Compensation Techniques Based on the Root-Locus Approach.

Case 2. $\gamma = \beta$. If $\gamma = \beta$ is required in Equation (7–6), then the preceeding design procedure for the lag–lead compensator may be modified as follows:

1. From the given performance specifications, determine the desired location for the dominant closed-loop poles.

2. The lag–lead compensator given by Equation (7–6) is modified to

$$G_c(s) = K_c \frac{(T_1 s + 1)(T_2 s + 1)}{\left(\frac{T_1}{\beta}s + 1\right)(\beta T_2 s + 1)} = K_c \frac{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{\beta}{T_1}\right)\left(s + \frac{1}{\beta T_2}\right)} \qquad (7\text{–}7)$$

where $\beta > 1$. The open-loop transfer function of the compensated system is $G_c(s)G(s)$. If the static velocity error constant K_v is specified, determine the value of constant K_c from the following equation:

$$K_v = \lim_{s\to 0} sG_c(s)G(s)$$

$$= \lim_{s\to 0} sK_cG(s)$$

Lag—lead Compensation Techniques Based on the Root-Locus Approach.

3. To have the dominant closed-loop poles at the desired location, calculate the angle contribution ϕ needed from the phase lead portion of the lag–lead compensator.

4. For the lag–lead compensator, we later choose T_2 sufficiently large so that

$$\left|\frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}}\right|$$

is approximately unity, where $s = s_1$ is one of the dominant closed-loop poles. Determine the values of T_1 and β from the magnitude and angle conditions:

$$\left|K_c\left(\frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\beta}{T_1}}\right)G(s_1)\right| = 1$$

$$\angle \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\beta}{T_1}} = \phi$$

8

Lag—lead Compensation Techniques Based on the Root-Locus Approach.

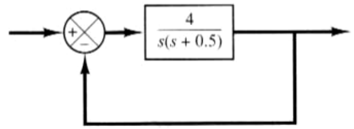
5. Using the value of β just determined, choose T_2 so that

$$\left|\frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}}\right| \doteq 1$$

$$-5^\circ < \angle \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} < 0^\circ$$

The value of βT_2, the largest time constant of the lag–lead compensator, should not be too large to be physically realized. (An example of the design of the lag–lead compensator when $\gamma = \beta$ is given in Example 7–4.)

Example1: Lag-Lead Compensator Case: $\gamma \neq \beta$



$$G(s) = \frac{4}{s(s+0.5)}$$

The closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G_c G(s)}{1+G_c G(s)} = \frac{4}{s^2 + 0.5s + 4}; \rightarrow \zeta = 0.125; \omega_n = 2 \text{ rad/sec}$$

Example1: Lag-Lead Compensator Case: $\gamma \neq \beta$

The dominant closed-loop poles are

$$s_{1,2} = -0.25 \pm j1.9843$$

The damping ratio $\zeta = 0.125$

The undamped natural frequency

$$\omega_n = 2 \text{ rad/sec}$$

The static velocity error constant is 8 sec^{-1}

Example1: Lag-Lead Compensator Case: $\gamma \neq \beta$

It is desired system of the dominant closed-loop poles

$\zeta = 0.5$ $\quad$ $\omega_n = 5 \text{ rad/sec}$

$$K_v = 80 \text{ sec}^{-1}$$

We use a lag-lead compensator having the transfer function

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right), \qquad (\gamma > 1, \beta > 1)$$

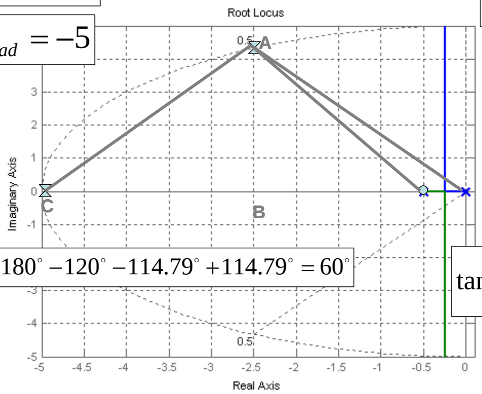
Example1: Lag-Lead Compensator Case: $\gamma \neq \beta$

$z_{lead} = -0.5$

$p_{lead} = -5$

$1 + K\dfrac{1}{s(s+0.5)} = 0$

$s_{1,2} = -2.5 \pm j4.33$



$\theta_c = 180^\circ - 120^\circ - 114.79^\circ + 114.79^\circ = 60^\circ$

$\tan 60^\circ = \dfrac{AB}{BC} = \dfrac{4.33}{BC}$

$BC = 2.5$

13

Example1: Lag-Lead Compensator Case: $\gamma \neq \beta$

Thus , the phase lead portion of the lag-lead compensator becomes

$$K_c \frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} = K_c \frac{s+0.5}{s+5}$$

where

$$T_1 = 2, \quad \gamma = \frac{5}{0.5} = 10$$

14

Example1: Lag-Lead Compensator Case: $\gamma \neq \beta$

Next we determine the value of K_c

$$\left| \frac{K_c(s+0.5)}{(s+5)} \frac{4}{(s+0.5)s} \right|_{s=-2.5+j4.33} = 1$$

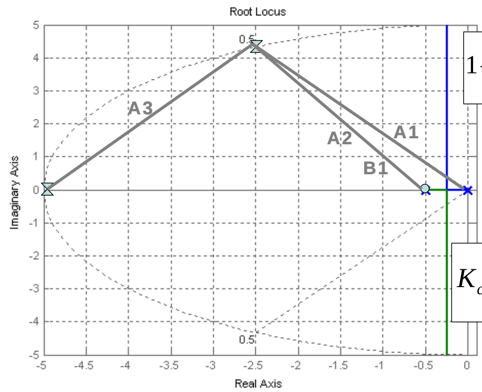
$$K_c = \frac{A_1 A_2 A_3}{4B_1} = \frac{5 \cdot 5}{4} = 6.25$$

Thus

$$K_c \frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} = 6.25 \frac{s+0.5}{s+5}$$

15

Example1: Lag-Lead Compensator Case: $\gamma \neq \beta$



$1 + K\dfrac{1}{s(s+0.5)} = 0$

$K_c = \dfrac{A_1 A_2 A_3}{4B_1} = 6.25$

16

Example1: Lag-Lead Compensator Case: $\gamma \neq \beta$

First the value of β is determined to satisfy the requirement on the static velocity error constant:

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} sK_c \frac{\beta}{\gamma} G(s)$$

$$K_v = \lim_{s\to 0} s(6.25)\frac{\beta}{10}\cdot\frac{4}{(s+0.5)s} = 5.008\beta = 80$$

Hence, β is determined as

$$\beta = 15.974$$

17

Example1: Lag-Lead Compensator Case: $\gamma \neq \beta$

Finally, we choose the value of T_2 large enough so that

$$\left|\frac{s+\frac{1}{T_2}}{s+\frac{1}{15.974T_2}}\right|_{s=-2.5+j4.33} = 1$$

and

$$-5^\circ < \angle\left.\frac{s+\frac{1}{T_2}}{s+\frac{1}{15.974T_2}}\right|_{s=-2.5+j4.33} < 0^\circ$$

We may choose $T_2 = 5$

18

Example1: Lag-Lead Compensator

Now the transfer function of the design lag-lead compensator is given by

$$G_c(s) = 6.25\left(\frac{s+0.5}{s+5}\right)\left(\frac{s+\frac{1}{5}}{s+\frac{1}{15.974\times 5}}\right)$$

$$G_c(s) = 6.25\left(\frac{s+0.5}{s+5}\right)\left(\frac{s+0.2}{s+0.0125}\right)$$

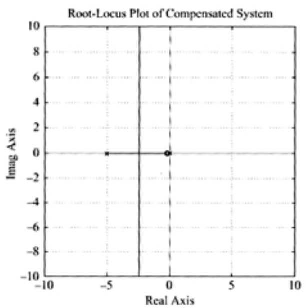
The compensated system will have the open-loop transfer function

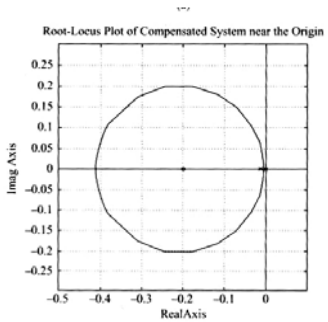
$$G_c(s)G(s) = 6.25\left(\frac{s+0.5}{s+5}\right)\left(\frac{s+0.2}{s+0.0125}\right)\left(\frac{4}{(s+0.5)s}\right)$$

19

Example1: Lag-Lead Compensator

The new damping ratio is ζ=0.491

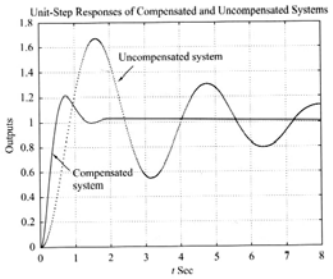


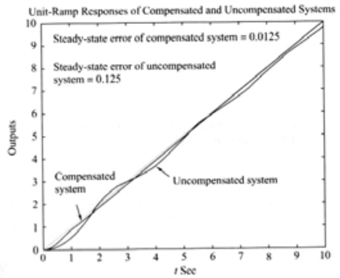


New closed-loop poles are located at s=-2.412±j4.275

20

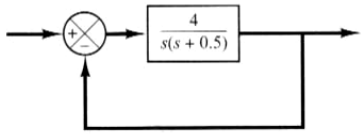
Example1: Lag-Lead Compensator





21

Example 2: Lag-Lead Compensator Case: γ = β



$$G(s)=\frac{4}{s(s+0.5)}$$

The closed-loop transfer function

$$\frac{C(s)}{R(s)}=\frac{G_cG(s)}{1+G_cG(s)}=\frac{4}{s^2+0.5s+4};\rightarrow \zeta=0.125;\omega_n=2\text{ rad/sec}$$

22

Example 2: Lag-Lead Compensator Case: γ = β

It is desired system of the dominant closed-loop poles

$$\zeta=0.5 \qquad \omega_n=5\text{ rad/sec}$$

$$K_v=80\ \text{sec}^{-1}$$

We use a lag-lead compensator having the transfer function

$$G_c(s)=K_c\left(\frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right),\qquad (\beta>1)$$

23

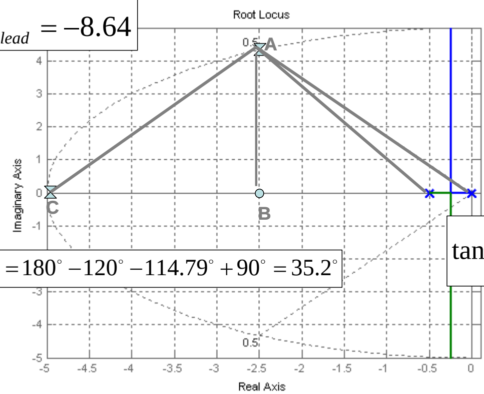
Example 2: Lag-Lead Compensator Case: γ = β

$$z_{lead}=-2.5$$

$$p_{lead}=-8.64$$

$$1+K\frac{1}{s(s+0.5)}=0$$

$$s_{1,2}=-2.5\pm j4.33$$



$$\theta_c=180^\circ-120^\circ-114.79^\circ+90^\circ=35.2^\circ$$

$$\tan 35.2^\circ=\frac{AB}{BC}=\frac{4.33}{BC}$$

$$BC=6.14$$

24

Example 2: Lag-Lead Compensator Case: γ = β

The phase lead portion of the lag-lead network thus become

$$z_{lead} = \frac{1}{T_1} = 2.5 \to T_1 = 0.4$$

$$p_{lead} = \frac{\beta}{T_1} = 8.64 \to \beta = 3.456$$

Since the requirement on the static velocity error constant is 80 sec^{-1}, We have

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} K_c \frac{4}{0.5} = 8K_c = 80 \to K_c = 10$$

Example 2: Lag-Lead Compensator Case: γ = β

For the phase lag portion, we may choose

$$T_2 = 10 \to z_{lag} = 0.1$$

$$p_{lag} = \frac{1}{\beta T_2} = \frac{1}{3.314 \times 10} = 0.03$$

Thus, the lag-lead compensator becomes

$$G_C(s) = 10\left(\frac{s+2.5}{s+8.64}\right)\left(\frac{s+0.1}{s+0.03}\right)$$

Example 2: Lag-Lead Compensator Case: γ = β

The compensated system will have the open-loop transfer function

$$G_c(s)G(s) = 10\left(\frac{s+2.5}{s+8.64}\right)\left(\frac{s+0.1}{s+0.03}\right)\left(\frac{4}{(s+0.5)s}\right)$$

The closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G_cG(s)}{1+G_cG(s)} = \frac{40(s^2+2.6s+0.25)}{s^4+8.69s^3+44.34s^2+104.122s+10}$$

New closed-loop poles and zeros are located at

$$p_1 = -3.968;\ p_2 = -0.1002;\ p_{3,4} = -2.3107 \pm j4.449$$

$$z_1 = -2.5;\ z_2 = -0.1$$

Example 2: Lag-Lead Compensator Case: γ = β

The steady-state error of the system for a unit-ramp input is

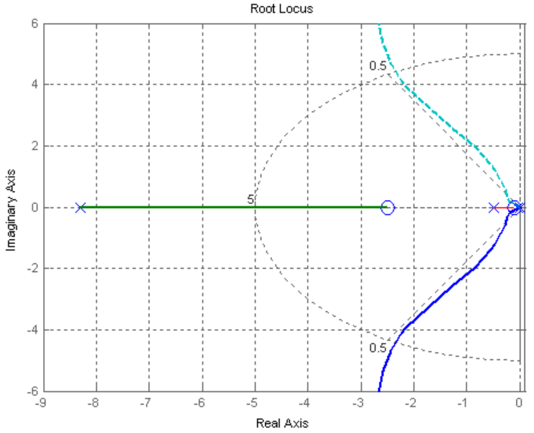
$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s^2}$$

$$= \lim_{s\to 0} \frac{(s+8.64)(s+0.03)(s+0.5)s}{s^4+8.69s^3+44.34s^2+104.122s+10} \cdot \frac{1}{s} = \frac{0.1296}{10} = 0.01296$$

The static velocity error constant, K_v is define by

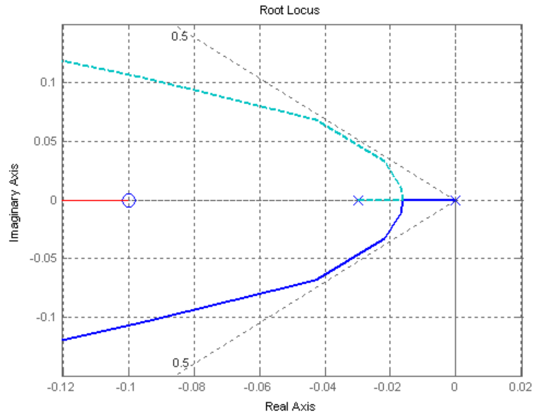
$$K_v = \frac{1}{e_{ss}} = \frac{1}{0.01296} = 77.16 \ \text{sec}^{-1}$$

Example 2: Lag-Lead Compensator Case: $\gamma = \beta$



29

Example 2: Lag-Lead Compensator Case: $\gamma = \beta$



30