8.0 INTRODUCTION

Cam-follower systems are frequently used in all kinds of machines. The valves in your automobile engine are opened by cams. Machines used in the manufacture of many consumer goods are full of cams. Compared to linkages, cams are easier to design to give a specific output function, but they are much more difficult and expensive to make than a linkage. Cams are a form of degenerate fourbar linkage in which the coupler link has been replaced by a half joint as shown in Figure 8-1. This topic was discussed in Section 2.9 (p. 40) on linkage transformation (see also Figure 2-10, p. 41). For anyone instantaneous position of earn and follower, we can substitute an effective linkage which will, for that instantaneous position, have the same motion as the original. In effect, the earn-follower is a fourbar linkage with variable-length (effective) links. It is this conceptual difference that makes the earn-follower such a flexible and useful function generator. We can specify virtually any output function we desire and quite likely create a curved surface on the earn to generate that function in the motion of the follower. We are not limited to fixed-length links as we were in linkage synthesis. The earn-follower is an extremely useful mechanical device, without which the machine designer's tasks would be more difficult to accomplish. But, as with everything else in engineering, there are trade-offs. These will be discussed in later sections. A list of the variables used in this chapter is provided in Table 8-1.

This chapter will present the proper approach to designing a earn-follower system, and in the process also present some less than proper designs as examples of the problems which inexperienced earn designers often get into. Theoretical considerations of the mathematical functions commonly used for earn curves will be discussed. Methods for the derivation of custom polynomial functions, to suit any set of boundary conditions, will be presented. The task of sizing the earn with considerations of pressure angle and radius of curvature will be addressed, and manufacturing processes and their limitations
discussed. The computer program DYNACAM will be used throughout the chapter as a tool to present and illustrate design concepts and solutions. A user manual for this program is in Appendix A. The reader can refer to that section at any time without loss of continuity in order to become familiar with the program’s operation.

8.1 CAM TERMINOLOGY

Cam-follower systems can be classified in several ways: by type of follower motion, either translating or rotating (oscillating); by type of cam, radial, cylindrical, three-dimensional; by type of joint closure, either force- or form-closed; by type of follower, curved or flat, rolling or sliding; by type of motion constraints, critical extreme position (CEP), critical path motion (CPM); by type of motion program, rise-fall (RF), rise-fall-dwell (RFD), rise-dwell-fall-dwell (RDFD). We will now discuss each of these classification schemes in more detail.

TABLE 8-1 Notation Used in This Chapter

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>time, seconds</td>
</tr>
<tr>
<td>$\theta$</td>
<td>camshaft angle, degrees or radians (rad)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>camshaft angular velocity, rad/sec</td>
</tr>
<tr>
<td>$\dot{\theta}$</td>
<td>total angle of any segment, rise, fall, or dwell, degrees or rad</td>
</tr>
<tr>
<td>$\dot{h}$</td>
<td>total lift (rise or fall) of any one segment, length units</td>
</tr>
<tr>
<td>$s$ or $S$</td>
<td>follower displacement, length units</td>
</tr>
<tr>
<td>$v = ds/d\theta$</td>
<td>follower velocity, length/rad</td>
</tr>
<tr>
<td>$V = dS/dt$</td>
<td>follower velocity, length/sec</td>
</tr>
<tr>
<td>$a = dv/d\theta$</td>
<td>follower acceleration, length/rad$^2$</td>
</tr>
<tr>
<td>$A = dv/dt$</td>
<td>follower acceleration, length/sec$^2$</td>
</tr>
<tr>
<td>$j = da/d\theta$</td>
<td>follower jerk, length/rad$^3$</td>
</tr>
<tr>
<td>$J = da/dt$</td>
<td>follower jerk, length/sec$^3$</td>
</tr>
<tr>
<td>$s, v, a, j$</td>
<td>refer to the group of diagrams, length units versus radians</td>
</tr>
<tr>
<td>$S, V, A, J$</td>
<td>refer to the group of diagrams, length units versus time</td>
</tr>
<tr>
<td>$R_b$</td>
<td>base circle radius, length units</td>
</tr>
<tr>
<td>$R_p$</td>
<td>prime circle radius, length units</td>
</tr>
<tr>
<td>$R_f$</td>
<td>roller follower radius, length units</td>
</tr>
<tr>
<td>$e$</td>
<td>eccentricity of cam-follower, length units</td>
</tr>
<tr>
<td>$\phi$</td>
<td>pressure angle, degrees or radians</td>
</tr>
<tr>
<td>$p$</td>
<td>radius of curvature of cam surface, length units</td>
</tr>
<tr>
<td>$p_{punc}$</td>
<td>radius of curvature of pitch curve, length units</td>
</tr>
<tr>
<td>$p_{min}$</td>
<td>minimum radius of curvature of pitch curve or cam surface, length units</td>
</tr>
</tbody>
</table>
Type of Follower Motion

Figure 8-1a shows a system with an oscillating, or rotating, follower. Figure 8-1b shows a translating follower. These are analogous to the crank-rocker fourbar and the slider-crank fourbar linkages, respectively. An effective fourbar linkage can be substituted for the cam-follower system for any instantaneous position. The lengths of the effective links are determined by the instantaneous locations of the centers of curvature of cam and follower as shown in Figure 8-1. The velocities and accelerations of the cam-follower system can be found by analyzing the behavior of the effective linkage for any position. A proof of this can be found in reference [1]. Of course, the effective links change length as the cam-follower moves giving it an advantage over a pure linkage as this allows more flexibility in meeting the desired motion constraints.

**Figure 8-1**

Effective linkages in the cam-follower
The choice between these two forms of the earn-follower is usually dictated by the type of output motion desired. If true rectilinear translation is required, then the translating follower is dictated. If pure rotation output is needed, then the oscillator is the obvious choice. There are advantages to each of these approaches, separate from their motion characteristics, depending on the type of follower chosen. These will be discussed in a later section.

**Type of Joint Closure**

*Force and form closure* were discussed in Section 2.3 (p. 24) on the subject of joints and have the same meaning here. *Force closure*, as shown in Figure 8-1 (p. 347), *requires an external force be applied to the joint* in order to keep the two links, cam and follower, physically in contact. This force is usually provided by a spring. This force, defined as positive in a direction which closes the joint, cannot be allowed to become negative. If it does, the links have lost contact because a force-closed joint can only push, not pull. *Form closure*, as shown in Figure 8-2, *closes the joint by geometry*. No external force is required. There are really two earn surfaces in this arrangement, one surface on each side of the follower. Each surface pushes, in its turn, to drive the follower in both directions.

Figure 8-2a and b shows track or groove cams which capture a single follower in the groove and both push and pull on the follower. Figure 8-2c shows another variety of form-closed earn-follower arrangement, called *conjugate* cams. There are two cams fixed on a common shaft which are mathematical conjugates of one another. Two roller followers, attached to a common arm, are each pushed in opposite directions by the conjugate cams. When form-closed cams are used in automobile or motorcycle engine valve trains, they are called *desmodromic* cams. There are advantages and disadvantages to both force- and form-closed arrangements which will be discussed in a later section.

**Type of Follower**

Follower, in this context, refers only to that part of the follower link which contacts the earn. Figure 8-3 shows three common arrangements, *flat-faced, mushroom* (curved), and *roller*. The roller follower has the advantage of lower (rolling) friction than the sliding contact of the other two but can be more expensive. *Flat-faced followers* can package smaller than roller followers for some earn designs and are often favored for that reason as well as cost for automotive valve trains. *Roller followers* are more frequently used in production machinery where their ease of replacement and availability from bearing manufacturers’ stock in any quantities are advantages. Grooved or track cams require roller followers. Roller followers are essentially ball or roller bearings with customized mounting details. Figure 8-5a shows two common types of commercial roller followers. Flat-faced or *mushroom followers* are usually custom designed and manufactured for each application. For high-volume applications such as automobile engines, the quantities are high enough to warrant a custom-designed follower.

**Type of Cam**

The direction of the follower's motion relative to the axis of rotation of the earn determines whether it is a *radial* or *axial* earn. All cams shown in Figures 8-1 to 8-3 are ra-
dial cams because the follower motion is generally in a radial direction. Open radial cams are also called plate cams.

Figure 8-4 shows an axial carn whose follower moves parallel to the axis of cam rotation. This arrangement is also called a face carn if open (force-closed) and a cylindrical or barrel carn if grooved or ribbed (form-closed).

Figure 8-5b shows a selection of cams of various types. Clockwise from the lower left, they are: an open (force-closed) axial or face carn; an axial grooved (track) carn (form-closed) with external gear; an open radial, or plate carn (force-closed); a ribbed axial carn (form-closed); an axial grooved (barrel) carn.
A three-dimensional cam or camoid (not shown) is a combination of radial and axial cams. It is a two-degree-of-freedom system. The two inputs are rotation of the cam about its axis and translation of the cam along its axis. The follower motion is a function of both inputs. The follower tracks along a different portion of the cam depending on the axial input.
Type of Motion Constraints

There are two general categories of motion constraint, critical extreme position (CEP; also called endpoint specification) and critical path motion (CPM). Critical extreme position refers to the case in which the design specifications define the start and finish positions of the follower (i.e., extreme positions) but do not specify any constraints on the path motion between the extreme positions. This case is discussed in Sections 8.3 and 8.4 and is the easier of the two to design as the designer has great freedom to choose the cam functions which control the motion between extremes. Critical path motion is a more constrained problem than CEP because the path motion, and/or one or more of its derivatives are defined over all or part of the interval of motion. This is analogous to function generation in the linkage design case except that with a cam we can achieve a continuous output function for the follower. Section 8.6 discusses this CPM case. It may only be possible to create an approximation of the specified function and still maintain suitable dynamic behavior.

Type of Motion Program

The motion programs rise-fall (RF), rise-fall-dwell (RFD), and rise-dwell-fall-dwell (RDFD) all refer mainly to the CEP case of motion constraint and in effect define how many dwells are present in the full cycle of motion, either none (RF), one (RFD), or more
than one (RDFD). **Dwells**, defined as no output motion for a specified period of input motion, are an important feature of cam-follower systems because it is very easy to create exact dwells in these mechanisms. The cam-follower is the design type of choice whenever a dwell is required. We saw in Section 3.9 (p. 125) how to design dwell linkages and found that at best we could obtain only an approximate dwell. The resulting single- or double-dwell linkages tend to be quite large for their output motion and are somewhat difficult to design. (See program SIXBAR for some built-in examples of these dwell linkages.) Cam-follower systems tend to be more compact than linkages for the same output motion.

If your need is for a **rise-fall** (RF) CEP motion, with no dwell, then you should really be considering a crank-rocker linkage rather than a cam-follower to obtain all the linkage's advantages over cams of reliability, ease of construction, and lower cost which were discussed in Section 2.15 (p. 55). If your needs for compactness outweigh those considerations, then the choice of a cam-follower in the RF case may be justified. Also, if you have a CPM design specification, and the motion or its derivatives are defined over the interval, then a cam-follower system is the logical choice in the RF case.

The **rise-fall-dwell** (RFD) and **rise-dwell-fall-dwell** (RDFD) cases are obvious choices for cam-followers for the reasons discussed above. However, each of these two cases has its own set of constraints on the behavior of the cam functions at the interfaces between the segments which control the rise, the fall, and the dwells. In general, we must match the **boundary conditions** (BCs) of the functions and their derivatives at all interfaces between the segments of the cam. This topic will be thoroughly discussed in the following sections.

### 8.2 S V A J DIAGRAMS

The first task faced by the cam designer is to select the mathematical functions to be used to define the motion of the follower. The easiest approach to this process is to "linearize" the cam, i.e., "unwrap it" from its circular shape and consider it as a function plotted on cartesian axes. We plot the displacement function \( s \), its first derivative velocity \( v \), its second derivative acceleration \( a \), and its third derivative jerk, all on aligned axes as a function of camshaft angle \( \theta \) as shown in Figure 8-6. Note that we can consider the independent variable in these plots to be either time \( t \) or shaft angle \( \theta \), as we know the constant angular velocity \( \omega \) of the camshaft and can easily convert from angle to time and vice versa.

\[
\dot{\theta} = \omega \tag{8.1}
\]

Figure 8-6a shows the specifications for a four-dwell cam that has eight segments, RDFDRDFD. Figure 8-6b shows the \( s v a j \) curves for the whole cam over 360 degrees of camshaft rotation. A cam design begins with a definition of the required cam functions and their \( s v a j \) diagrams. Functions for the nondwell cam segments should be chosen based on their velocity, acceleration, and jerk characteristics and the relationships at the interfaces between adjacent segments including the dwells. These function characteristics can be conveniently and quickly investigated with program DYNACAM which generated the data and plots shown in Figure 8-6.
8.3 DOUBLE-DWELL CAM DESIGN-CHOOSING FUNCTIONS

Many cam design applications require multiple dwells. The double-dwell case is quite common. Perhaps a double-dwell cam is driving a part feeding station on a production machine that makes toothpaste. This hypothetical cam's follower is fed an empty toothpaste tube (during the low dwell), then moves the empty tube into a loading station (during the rise), holds the tube absolutely still in a critical extreme position (CEP) while toothpaste is squirted into the open bottom of the tube (during the high dwell), and then retracts the filled tube back to the starting (zero) position and holds it in this other critical extreme position. At this point, another mechanism (during the low dwell) picks the tube up and carries it to the next operation, which might be to seal the bottom of the tube. A similar cam could be used to feed, align, and retract the tube at the bottom-sealing station as well.

Cam specifications such as this are often depicted on a timing diagram as shown in Figure 8-7 which is a graphical representation of the specified events in the machine cycle. A machine's cycle is defined as one revolution of its master driveshaft. In a complicated machine, such as our toothpaste maker, there will be a timing diagram for each subassembly in the machine. The time relationships among all subassemblies are defined by their timing diagrams which are all drawn on a common time axis. Obviously all these operations must be kept in precise synchrony and time phase for the machine to work.
Solution:

I The naive or inexperienced cam designer might proceed with a design as shown in Figure 8-8a. Taking the given specifications literally, it is tempting to merely "connect the dots" on the timing diagram to create the displacement (s) diagram. (After all, when we wrap this s diagram around a circle to create the actual cam, it will look quite smooth despite the sharp corners on the s diagram.) The mistake our beginning designer is making here is to ignore the effect on the higher derivatives of the displacement function which results from this simplistic approach.

Figure 8-8b, c, and d shows the problem. Note that we have to treat each segment of the cam (rise, fall, dwell) as a separate entity in developing mathematical functions for the cam. Taking the rise segment (#2) first, the displacement function in Figure 8-8a during this portion is a straight line, or first-degree polynomial. The general equation for a straight line is:

\[ s = mt + b \]
where \( m \) is the slope of the line and \( b \) is the \( y \) intercept. Substituting variables appropriate to this example in equation 8.2, angle \( \theta \) replaces the independent variable \( x \), and the displacement \( s \) replaces the dependent variable \( y \). By definition, the constant slope \( m \) of the displacement is the velocity constant \( K_v \).

3 For the rise segment, the \( y \) intercept \( b \) is zero because the low dwell position typically is taken as zero displacement by convention. Equation 8.2 then becomes:

\[
    s = K_v \theta 
\]  

(8.3)

4 Differentiating with respect to \( \theta \) gives a function for velocity during the rise.

\[
    v = K_v = \text{constant} 
\]

(8.4)

5 Differentiating again with respect to \( \theta \) gives a function for acceleration during the rise.

\[
    a = 0 
\]

(8.5)
This seems too good to be true (and it is). Zero acceleration means zero dynamic force. This cam appears to have no dynamic forces or stresses in it!

Figure 8-8 shows what is really happening here. If we return to the displacement function and graphically differentiate it twice, we will observe that, from the definition of the derivative as the instantaneous slope of the function, the acceleration is in fact zero during the interval. **But, at the boundaries of the interval, where rise meets low dwell on one side and high dwell on the other, note that the velocity function is multivalued. There are discontinuities at these boundaries.** The effect of these discontinuities is to create a portion of the velocity curve which has infinite slope and zero duration. This results in the **infinite spikes of acceleration** shown at those points.

These spikes are more properly called Dirac delta functions. Infinite acceleration cannot really be obtained, as it requires infinite force. Clearly the dynamic forces will be very large at these boundaries and will create high stresses and rapid wear. In fact, if this cam were built and run at any significant speeds, the sharp corners on the displacement diagram which are creating these theoretical infinite accelerations would be quickly worn to a smoother contour by the unsustainable stresses generated in the materials. This is an unacceptable design.

The unacceptability of this design is reinforced by the jerk diagram which shows theoretical values of infinity squared at the discontinuities. The problem has been engendered by an inappropriate choice of displacement function. In fact, the cam designer should not be as concerned with the displacement function as with its higher derivatives.

**The Fundamental law of Cam Design**

Any cam designed for operation at other than very low speeds must be designed with the following constraints:

*The cam function must be continuous through the first and second derivatives of displacement across the entire interval (360 degrees).*

**corollary:**

*The jerk function must be finite across the entire interval (360 degrees).*

In any but the simplest of cams, the cam motion program cannot be defined by a single mathematical expression, but rather must be defined by several separate functions, each of which defines the follower behavior over one segment, or piece, of the cam. These expressions are sometimes called **piecewise functions.** These functions must have third-order continuity (the function plus two derivatives) at all boundaries. The displacement, velocity and acceleration functions must have no discontinuities in them.

*If any discontinuities exist in the acceleration function, then there will be infinite spikes, or Dirac delta functions, appearing in the derivative of acceleration, jerk. Thus the corollary merely restates the fundamental law of cam design. Our naive designer failed to recognize that by starting with a low-degree (linear) polynomial as the displacement function, discontinuities would appear in the upper derivatives.*

Polynomial functions are one of the best choices for cams as we shall shortly see, but they do have one fault that can lead to trouble in this application. Each time they are
differentiated, they reduce by one degree. Eventually, after enough differentiations, polynomials degenerate to zero degree (a constant value) as the velocity function in Figure 8-8b (p. 355) shows. Thus, by starting with a first-degree polynomial as a displacement function, it was inevitable that discontinuities would soon appear in its derivatives.

In order to obey the fundamental law of cam design, one must start with at least a third-degree polynomial (cubic) as the displacement function. This will degenerate to a first-degree function in the acceleration. The jerk function will have discontinuities, and the (unnamed) derivative of jerk will have infinite spikes in it. This is acceptable, as the jerk is still finite.

**Simple Harmonic Motion (SHM)**

Our naïve cam designer recognized his mistake in choosing a straight-line function for the displacement. He also remembered a family of functions he had met in a calculus course which have the property of remaining continuous throughout any number of differentiations. These are the harmonic functions. On repeated differentiation, sine becomes cosine, which becomes negative sine, which becomes negative cosine, etc., ad infinitum. One never runs out of derivatives with the harmonic family of curves. In fact, differentiation of a harmonic function really only amounts to a 90° phase shift of the function. It is though, as you differentiated it, you cut out, with a scissors, a different portion of the same continuous sine wave function, which is defined from minus infinity to plus infinity. The equations of simple harmonic motion (SHM) for a rise motion are:

\[
\begin{align*}
    s &= \frac{h}{2} \left[ 1 - \cos \left( \frac{\pi \theta}{\beta} \right) \right] \\
    v &= \frac{\pi h}{\beta} \sin \left( \frac{\pi \theta}{\beta} \right) \\
    a &= \frac{\pi^2 h}{\beta^2} \cos \left( \frac{\pi \theta}{\beta} \right) \\
    j &= -\frac{\pi^3 h}{\beta^3} \sin \left( \frac{\pi \theta}{\beta} \right)
\end{align*}
\]  

(8.6a)

(8.6b)

(8.6c)

(8.6d)

where \( h \) is the total rise, or lift, \( \theta \) is the camshaft angle, and \( \beta \) is the total angle of the rise interval.

We have here introduced a notation to simplify the expressions. The independent variable in our cam functions is \( \theta \), the camshaft angle. The period of any one segment is defined as the angle \( \beta \). Its value can, of course, be different for each segment. We normalize the independent variable \( \theta \) by dividing it by the period of the segment \( \beta \). Both \( \theta \) and \( \beta \) are measured in radians (or both in degrees). The value of \( \theta/\beta \) will then vary from 0 to 1 over any segment. It is a dimensionless ratio. Equations 8.6 define simple harmonic motion and its derivatives for this rise segment in terms of \( \theta/\beta \).

This family of harmonic functions appears, at first glance, to be well suited to our cam design problem above. If we define the displacement function to be one of the harmonic functions, we should not “run out of derivatives” before reaching the acceleration.
EXAMPLE 8-2

Sophomoric* Cam Design—Simple Harmonic Motion—Still a Bad Cam.

Problem:  Consider the same cam design CEP specification as in Example 8-1:

- **dwell** at zero displacement for 90 degrees (low dwell)
- **rise** 1 in (25 mm) in 90 degrees
- **dwell** at 1 in (25 mm) for 90 degrees (high dwell)
- **fail** 1 in (25 mm) in 90 degrees
- **cam** 2π rad/sec = 1 rev/sec

Solution:

1. Figure 8-9 shows a full-rise simple harmonic function† applied to the rise segment of our cam design problem.

2. Note that the velocity function is continuous, as it matches the zero velocity of the dwells at each end. The peak value is 6.28 in/sec (160 mm/sec) at the midpoint of the rise.

3. The acceleration function, however, is not continuous. It is a half-period cosine curve and has nonzero values at start and finish which are ± 7.8 m/sec².

4. Unfortunately, the dwell functions, which adjoin this rise on each side, have zero acceleration as can be seen in Figure 8-6 (p. 253). Thus there are discontinuities in the acceleration at each end of the interval which uses this simple harmonic displacement function.

5. This violates the fundamental law of cam design and creates infinite spikes of jerk at the ends of this fall interval. This is also an unacceptable design.

---

* Sophomoric, from sophomore. *Sopho* wise fool, from the Greek, sophos = wisdom, moros = fool.

† Though this is actually a half-period cosine wave, we will call it a full-rise (or full-fall) simple harmonic function to differentiate it from the half-rise (and half-fall) simple harmonic function which is actually a quarter-period cosine (see Section 8.6, p. 385).

---

FIGURE 8-9

Simple harmonic motion with dwells has discontinuous acceleration
What went wrong? While it is true that harmonic functions are differentiable ad infinitum, we are not dealing here with single harmonic functions. Our cam function over the entire interval is a piecewise function, (Figure 8-6, p. 353) made up of several segments, some of which may be dwell portions or other functions. A dwell will always have zero velocity and zero acceleration. Thus we must match the dwells' zero values at the ends of those derivatives of any nondwell segments that adjoin them. The simple harmonic displacement function, when used with dwells, does not satisfy the fundamental law of cam design. Its second derivative, acceleration, is nonzero at its ends and thus does not match the dwells required in this example.

The only case in which the simple harmonic displacement function will satisfy the fundamental law is the non-quick-return RF case, i.e., rise in 180° and fall in 180° with no dwells. Then the cam becomes an eccentric as shown in Figure 8-10. As a single continuous (not piecewise) function, its derivatives are continuous also. Figure 8-11 shows the displacement (in inches) and acceleration functions (in g’s) of the eccentric cam in Figure 8-10 as actually measured on the follower. The noise, or “ripple,” on the acceleration curve is due to small, unavoidable, manufacturing errors. Manufacturing limitations will be discussed in a later section.

Cycloidal Displacement

The two bad examples of cam design described above should lead the cam designer to the conclusion that consideration only of the displacement function when designing a cam is erroneous. The better approach is to start with consideration of the higher derivatives, especially acceleration. The acceleration function, and to a lesser extent the jerk
function, should be the principal concern of the designer. In some cases, especially when the mass of the follower train is large, or when there is a specification on velocity, that function must be carefully designed as well.

With this in mind, we will redesign the cam for the same example specifications as above. This time we will start with the acceleration function. The harmonic family of functions still have advantages which make them attractive for these applications. Figure 8-12 shows a full-period sinusoid applied as the acceleration function. It meets the constraint of zero magnitude at each end to match the dwell segments which adjoin it. The equation for a sine wave is:

$$a = C \sin \left( \frac{2\pi \theta}{\beta} \right)$$  \hspace{1cm} (8.7)

We have again normalized the independent variable $\theta$ by dividing it by the period of the segment $\beta$ with both $\theta$ and $\beta$ measured in radians. The value of $\theta/\beta$ ranges from 0 to 1 over any segment and is a dimensionless ratio. Since we want a full-cycle sine wave, we must multiply the argument by $2\pi$. The argument of the sine function will then vary between 0 and $2\pi$ regardless of the value of $\beta$. The constant $C$ defines the amplitude of the sine wave.

Integrate to obtain velocity,

$$a = \frac{dv}{d\theta} = C \sin \left( \frac{2\pi \theta}{\beta} \right)$$

$$\int dv = \int C \sin \left( \frac{2\pi \theta}{\beta} \right) d\theta$$

$$v = -C \frac{\beta}{2\pi} \cos \left( \frac{2\pi \theta}{\beta} \right) + k_1$$  \hspace{1cm} (8.8)

**FIGURE 8-12**
Sinusoidal acceleration gives cycloidal displacement
where \( k_1 \) is the constant of integration. To evaluate \( k_1 \), substitute the boundary condition \( v = 0 \) at \( \Theta = 0 \), since we must match the zero velocity of the dwell at that point. The constant of integration is then:

\[
k_1 = C \frac{\beta}{2\pi}
\]

and:

\[
v = C \frac{\beta}{2\pi} \left[ 1 - \cos \left( \frac{2\pi \Theta}{\beta} \right) \right]
\]  

(8.9)

Note that substituting the boundary values at the other end of the interval, \( v = 0, \Theta = \beta \), will give the same result for \( k_1 \). Integrate again to obtain displacement:

\[
v = \frac{ds}{d\Theta} = C \frac{\beta}{2\pi} \left[ 1 - \cos \left( \frac{2\pi \Theta}{\beta} \right) \right]
\]

\[
\int ds = \left[ C \frac{\beta}{2\pi} \left[ 1 - \cos \left( \frac{2\pi \Theta}{\beta} \right) \right] \right] d\Theta
\]

\[
s = C \frac{\beta}{2\pi} \Theta - C \frac{\beta^2}{4\pi^2} \sin \left( \frac{2\pi \Theta}{\beta} \right) + k_2
\]  

(8.10)

To evaluate \( k_2 \), substitute the boundary condition \( s = 0 \) at \( \Theta = 0 \), since we must match the zero displacement of the dwell at that point. To evaluate the amplitude constant \( C \), substitute the boundary condition \( s = h \) at \( \Theta = \beta \), where \( h \) is the maximum follower rise (or lift) required over the interval and is a constant for any one cam specification.

\[
k_2 = 0
\]

\[
C = 2\pi \frac{h}{\beta^2}
\]  

(8.11)

Substituting the value of the constant \( C \) in equation 8.7 for acceleration gives:

\[
a = 2\pi \frac{h}{\beta^2} \sin \left( \frac{2\pi \Theta}{\beta} \right)
\]  

(8.12a)

Differentiating with respect to \( \Theta \) gives the expression for jerk.

\[
j = 4\pi^2 \frac{h}{\beta^3} \cos \left( \frac{2\pi \Theta}{\beta} \right)
\]  

(8.12b)

Substituting the values of the constants \( C \) and \( k_1 \) in equation 8.9 for velocity gives:

\[
v = \frac{h}{\beta} \left[ 1 - \cos \left( \frac{2\pi \Theta}{\beta} \right) \right]
\]  

(8.12c)

This velocity function is the sum of a negative cosine term and a constant term. The coefficient of the cosine term is equal to the constant term. This results in a velocity curve which starts and ends at zero and reaches a maximum magnitude at \( \beta/2 \) as can be
seen in Figure 8-12. Substituting the values of the constants $C$, $k_1$, and $k_2$ in equation 8.10 for displacement gives:

$$s = h \left[ \frac{\theta}{\beta} - \frac{1}{2\pi} \sin \left( 2\pi \frac{\theta}{\beta} \right) \right]$$  \hspace{1cm} (8.12d)

Note that this displacement expression is the sum of a straight line of slope $h$ and a negative sine wave. The sine wave is, in effect, "wrapped around" the straight line as can be seen in Figure 8-12. Equation 8.12d is the expression for a cycloid. This cam function is referred to either as **cycloidal displacement** or **sinusoidal acceleration**.

In the form presented, with $\theta$ (in radians) as the independent variable, the units of equation 8.12d are length, of equation 8.12e length/rad, of equation 8.12a length/rad$^2$, and of equation 8.12b length/rad$^3$. To convert these equations to a time base, multiply velocity $v$ by the camshaft angular velocity $\omega$ (in rad/sec), multiply acceleration $a$ by $\omega^2$, and jerk $j$ by $\omega^3$.

**EXAMPLE 8-3**

Junior Cam Design—Cycloidal Displacement—An Acceptable Cam.

**Problem:** Consider the same cam design CEP specification as in Examples 8-1 and 8-2:

- **dwell** at zero displacement for 90 degrees (low dwell)
- **rise** 1 in (25 mm) in 90 degrees
- **dwell** at 1 in (25 mm) for 90 degrees (high dwell)
- **fall** 1 in (25 mm) in 90 degrees
- **cam \( \omega \)** 2\(\pi\) rad/sec = 1 rev/sec

**Solution:**

1. The cycloidal displacement function is an acceptable one for this double-dwell cam specification. Its derivatives are continuous through the acceleration function as seen in Figure 8-12. The peak acceleration is 100.4 in/sec$^2$ (2.55 m/sec$^2$).

2. The jerk curve in Figure 8-12 is discontinuous at its boundaries but is of finite magnitude, and this is acceptable. Its peak value is 2523 in/sec$^3$ (64 m/sec$^3$).

3. The velocity is smooth and matches the zeros of the dwell at each end. Its peak value is 8 in/sec (0.2 m/sec).

4. The only drawback to this function is that it has relatively large magnitudes of peak acceleration and peak velocity compared to some other possible functions for the double-dwell case.

The reader may input the file E08-02.cam to program **DYNACAM** to investigate this example in more detail.

**Combined Functions**

Dynamic force is proportional to acceleration. We generally would like to minimize dynamic forces, and thus should be looking to minimize the magnitude of the accelerat-
tion function as well as to keep it continuous. Kinetic energy is proportional to velocity squared. We also would like to minimize stored kinetic energy, especially with large mass follower trains, and so are concerned with the magnitude of the velocity function as well.

**CONSTANT ACCELERATION** If we wish to minimize the peak value of the magnitude of the acceleration function for a given problem, the function that would best satisfy this constraint is the square wave as shown in Figure 8-13. This function is also called **constant acceleration**. The square wave has the property of minimum peak value for a given area in a given interval. However, this function is not continuous. It has discontinuities at the beginning, middle, and end of the interval, so, by itself, this is unacceptable as a cam acceleration function.

**TRAPEZOIDAL ACCELERATION** The square wave’s discontinuities can be removed by simply “knocking the corners off” the square wave function and creating the **trapezoidal acceleration** function shown in Figure 8-14a. The area lost from the “knocked off corners” must be replaced by increasing the peak magnitude above that of the original square wave in order to maintain the required specifications on lift and duration. But, this increase in peak magnitude is small, and the theoretical maximum acceleration can be significantly less than the theoretical peak value of the sinusoidal acceleration (cycloidal displacement) function. One disadvantage of this trapezoidal function is its very discontinuous jerk function, as shown in Figure 8-14b. Ragged jerk functions such as this tend to excite vibratory behavior in the follower train due to their high harmonic content. The cycloidal’s sinusoidal acceleration has a relatively smoother cosine jerk function with only two discontinuities in the interval and is preferable to the trapezoid’s square waves of jerk. But the cycloidal’s theoretical peak acceleration will be larger, which is not desirable. So, trade-offs must be made in selecting the cam functions.

![Figure 8-13](image)

**FIGURE 8-13**
Constant acceleration gives infinite jerk
MODIFIED TRAPEZOIDAL ACCELERATION  An improvement can be made to the trapezoidal acceleration function by substituting pieces of sine waves for the sloped sides of the trapezoids as shown in Figure 8-15. This function is called the modified trapezoidal acceleration curve.* This function is a marriage of the sine acceleration and constant acceleration curves. Conceptually, a full period sine wave is cut into fourths and "pasted into" the square wave to provide a smooth transition from the zeros at the endpoints to the maximum and minimum peak values, and to make the transition from maximum to minimum in the center of the interval. The portions of the total segment period ($\beta$) used for the sinusoidal parts of the function can be varied. The most common arrangement is to cut the square wave at $\beta/8, 3\beta/8, 5\beta/8, \text{and } 7\beta/8$ to insert the pieces of sine wave as shown in Figure 8-15. The $s v a j$ formulas for that arrangement of a modified trapezoidal rise are:

\[
\begin{align*}
\theta &< \frac{1}{8} \beta \\
0 &< \theta \leq \frac{1}{8} \beta \\
\theta &> \frac{1}{8} \beta \\
\theta &> \frac{3}{8} \beta \\
\theta &> \frac{5}{8} \beta \\
\theta &> \frac{7}{8} \beta \\
\end{align*}
\]

For $0 \leq \theta < \frac{1}{8} \beta$:

\[
\begin{align*}
\dot{s} &= h \left[ 0.38898448 \frac{\theta}{\beta} - 0.0309544 \sin \left( 4\pi \frac{\theta}{\beta} \right) \right] \\
\dot{v} &= 0.38898448 \frac{h}{\beta} \left[ 1 - \cos \left( 4\pi \frac{\theta}{\beta} \right) \right] \\
\dot{a} &= 4.888124 \frac{h}{\beta^2} \sin \left( 4\pi \frac{\theta}{\beta} \right) \\
\dot{j} &= 61.425769 \frac{h}{\beta^3} \cos \left( 4\pi \frac{\theta}{\beta} \right)
\end{align*}
\]

---

* Developed by C. N. Neklutan of the Universal Match Corp. See reference [2].
for $\frac{1}{8} \beta \leq \theta < \frac{3}{8} \beta$

$$s = h \left[ 2.44406184 \left( \frac{\theta}{\beta} \right)^2 - 0.22203097 \left( \frac{\theta}{\beta} \right) + 0.00723407 \right]$$

$$v = \frac{h}{\beta} \left[ 4.888124 \left( \frac{\theta}{\beta} \right)^2 - 0.22203097 \right]$$

$$a = 4.888124 \frac{h}{\beta^2}$$

$$j = 0$$

for $\frac{3}{8} \beta \leq \theta < \frac{5}{8} \beta$

$$s = h \left[ 1.6110154 \frac{\theta}{\beta} - 0.0309544 \sin \left( 4\pi \frac{\theta}{\beta} - \pi \right) - 0.3055077 \right]$$

$$v = \frac{h}{\beta} \left[ 1.6110154 - 0.38398448 \cos \left( 4\pi \frac{\theta}{\beta} - \pi \right) \right]$$

$$a = 4.888124 \frac{h}{\beta^2} \sin \left( 4\pi \frac{\theta}{\beta} - \pi \right)$$

$$j = 61.425769 \frac{h}{\beta^3} \cos \left( 4\pi \frac{\theta}{\beta} - \pi \right)$$

for $\frac{5}{8} \beta \leq \theta < \frac{7}{8} \beta$

$$s = h \left[ -2.44406184 \left( \frac{\theta}{\beta} \right)^2 + 4.6669091 \left( \frac{\theta}{\beta} \right) - 1.2292648 \right]$$

$$v = \frac{h}{\beta} \left[ -4.888124 \left( \frac{\theta}{\beta} \right) + 4.6669017 \right]$$

$$a = -4.888124 \frac{h}{\beta^2}$$

$$j = 0$$

for $\frac{7}{8} \beta \leq \theta \leq \beta$

$$s = h \left[ 0.6120154 + 0.38398448 \frac{\theta}{\beta} + 0.0309544 \sin \left( 4\pi \frac{\theta}{\beta} - 3\pi \right) \right]$$

$$v = 0.38398448 \frac{h}{\beta} 1 + \cos \left( 4\pi \frac{\theta}{\beta} - 3\pi \right)$$

$$a = -4.888124 \frac{h}{\beta^2} \sin \left( 4\pi \frac{\theta}{\beta} - 3\pi \right)$$

$$j = -61.425769 \frac{h}{\beta^3} \cos \left( 4\pi \frac{\theta}{\beta} - 3\pi \right)$$
FIGURE 8-15
Creating the modified trapezoidal acceleration function
The modified trapezoidal function defined above is one of many combined functions created for cams by piecing together various functions, while being careful to match the values of the $s$, $v$, and $a$ curves at all the interfaces between the joined functions. It has the advantage of relatively low theoretical peak acceleration, and reasonably rapid, smooth transitions at the beginning and end of the interval. Note that in the form presented, with $\theta$ (in radians) as the independent variable, the units of the expressions in equations 8.13 are length, length/rad, length/rad$^2$, and length/rad$^3$ for $s$, $v$, $a$, $j$, respectively. To convert equations 8.13 to a time base, multiply velocity $v$ by the camshaft angular velocity $\omega$ (in rad/sec), acceleration $a$ by $\omega^2$, and jerk $j$ by $\omega^3$. The modified trapezoidal cam function is a popular and often used program for double-dwell cams. Its $s$, $v$, $a$, $j$ curves are shown in Figure 8-16.

**MODIFIED SINUSOIDAL ACCELERATION**  

The sine acceleration curve (cycloidal displacement) has the advantage of smoothness (less ragged jerk curve) compared to the modified trapezoid but has higher theoretical peak acceleration. By combining two harmonic (sinusoid) curves of different frequencies, we can retain some of the smoothness characteristics of the cycloid and also reduce the peak acceleration. As an added bonus we will find that the peak velocity is also lower than in either the cycloidal or modified trapezoid. Figure 8-17 shows how the modified sine acceleration curve is made up of pieces of two sinusoid functions, one of higher frequency than the other. The first and last quarter of the high-frequency (short period, $\beta/2$) sine curve is used for the first and last eights of the combined function. The center half of the low-frequency (long period, $3\beta/2$) sine wave is used to fill in the center three-fourths of the combined curve. Obviously, the magnitudes of the two curves and their derivatives must be matched at their interfaces in order to avoid discontinuities. The equations for the modified sine curve for a rise of height $h$ over a period $\beta$, with the functions joined at the $\beta/8$ and $7\beta/8$ points are as follows:

---

*Developed by E. H. Schmidt of DuPont.*
FIGURE 8-17
Creating the modified sine acceleration function
for \(0 \leq \theta < \frac{1}{8} \beta\)

\[
s = h \left[ 0.43990085 \frac{\theta}{\beta} - 0.0350062 \sin \left( 4\pi \frac{\theta}{\beta} \right) \right]
\]

\[
\nu = 0.43990085 \frac{h}{\beta^2} \left[ 1 - \cos \left( 4\pi \frac{\theta}{\beta} \right) \right]
\]

\[
a = 5.527957 \frac{h}{\beta^2} \sin \left( 4\pi \frac{\theta}{\beta} \right)
\]

\[
j = 69.4663577 \frac{h}{\beta^3} \cos \left( 4\pi \frac{\theta}{\beta} \right)
\]

for \(\frac{1}{8} \beta \leq \theta < \frac{7}{8} \beta\)

\[
s = h \left[ 0.28004957 + 0.43990085 \frac{\theta}{\beta} - 0.31505577 \cos \left( \frac{4\pi \theta}{3 \beta} - \frac{\pi}{6} \right) \right]
\]

\[
\nu = 0.43990085 \frac{h}{\beta^2} \left[ 1 + 3 \sin \left( \frac{4\pi \theta}{3 \beta} - \frac{\pi}{6} \right) \right]
\]

\[
a = 5.527957 \frac{h}{\beta^2} \cos \left( \frac{4\pi \theta}{3 \beta} - \frac{\pi}{6} \right)
\]

\[
j = -23.1553 \frac{h}{\beta^3} \sin \left( \frac{4\pi \theta}{3 \beta} - \frac{\pi}{6} \right)
\]

for \(\frac{7}{8} \beta \leq \theta \leq \beta\)

\[
s = h \left[ 0.5669915 + 0.43990085 \frac{\theta}{\beta} - 0.0350062 \sin \left( 2\pi \left( \frac{2 \theta}{\beta} - 1 \right) \right) \right]
\]

\[
\nu = 0.43990085 \frac{h}{\beta^2} \left[ 1 - \cos \left( 2\pi \left( \frac{2 \theta}{\beta} - 1 \right) \right) \right]
\]

\[
a = 5.527957 \frac{h}{\beta^2} \sin \left( 2\pi \left( \frac{2 \theta}{\beta} - 1 \right) \right)
\]

\[
j = 69.4663577 \frac{h}{\beta^3} \cos \left( 2\pi \left( \frac{2 \theta}{\beta} - 1 \right) \right)
\]

Figure 8-18 shows a comparison of the shapes and relative magnitudes of five cam acceleration programs including the cycloidal, modified trapezoid, and modified sine acceleration curves. The cycloidal curve has a theoretical peak acceleration which is approximately 1.3 times that of the modified trapezoid's peak value for the same cam position.* The 3-4-5 and 4-5-6-7 polynomial functions also shown in the figure will be discussed in a later section.
specification. The peak value of acceleration for the modified sine is between the cycloidal and modified trapezoid. Table 8-2 lists the peak values of acceleration, velocity, and jerk for these functions in terms of the total rise $h$ and period $\beta$.

Figure 8-19 compares the jerk curves for the same functions. The modified sine jerk is somewhat less ragged than the modified trapezoid jerk but not as smooth as that of the cycloid, which is a full-period cosine. Figure 8-20 compares their velocity curves. The peak velocities of the cycloidal and modified trapezoid functions are the same, so each will store the same peak kinetic energy in the follower train. The peak velocity of the modified sine is the lowest of the five functions shown. This is the principal advantage of the modified sine acceleration curve and the reason it is usually chosen for applications in which the follower mass is very large.

### TABLE 8-2  Factors for Peak Velocity and Acceleration of Some Cam Functions

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant accel.</td>
<td>2.000 $h/\beta$</td>
<td>4.000 $h/\beta^2$</td>
<td>infinite</td>
<td>$\infty$ jerk - not acceptable.</td>
</tr>
<tr>
<td>Harmonic disp.</td>
<td>1.571 $h/\beta$</td>
<td>4.945 $h/\beta^2$</td>
<td>infinite</td>
<td>$\infty$ jerk - not acceptable.</td>
</tr>
<tr>
<td>Trapezoid accel.</td>
<td>2.000 $h/\beta$</td>
<td>5.300 $h/\beta^2$</td>
<td>$44 h/\beta^3$</td>
<td>Not as good as mod. trap.</td>
</tr>
<tr>
<td>Mod. trap. accel.</td>
<td>2.000 $h/\beta$</td>
<td>4.888 $h/\beta^2$</td>
<td>$61 h/\beta^3$</td>
<td>Low accel but rough jerk.</td>
</tr>
<tr>
<td>Mod. sine. accel.</td>
<td>1.760 $h/\beta$</td>
<td>5.528 $h/\beta^2$</td>
<td>$69 h/\beta^3$</td>
<td>Low veloc - good accel.</td>
</tr>
<tr>
<td>3-4-5 Poly. disp.</td>
<td>1.875 $h/\beta$</td>
<td>5.777 $h/\beta^2$</td>
<td>$60 h/\beta^3$</td>
<td>Good compromise.</td>
</tr>
<tr>
<td>Cycloidal disp.</td>
<td>2.000 $h/\beta$</td>
<td>6.283 $h/\beta^2$</td>
<td>$40 h/\beta^3$</td>
<td>Smooth accel. &amp; jerk.</td>
</tr>
<tr>
<td>4-5-6-7 Poly. disp.</td>
<td>2.188 $h/\beta$</td>
<td>7.526 $h/\beta^2$</td>
<td>$52 h/\beta^3$</td>
<td>Smooth jerk-high accel.</td>
</tr>
</tbody>
</table>
An example of such an application is shown in Figure 8-21 which is an indexing table drive used for automated assembly lines. The round indexing table is mounted on
Figure 8.21
Multi-stop rotary indexer. Courtesy of The Ferguson Co., St. Louis, MO.

A tapered vertical spindle and driven as part of the follower train by a form-closed barrel cam which moves it through some angular displacement, and then holds the table still in a dwell (called a "stop") while an assembly operation is performed on the workpiece carried on the table. These indexers may have three or more stops, each corresponding to an index position. The table (not shown) is solid steel and may be several feet in diameter; thus its mass is large. To minimize the stored kinetic energy, which must be dissipated each time the table is brought to a stop, the manufacturers often use the modified sine program on these multidwell cams, because of its lower peak velocity.

Let us again try to improve the double-dwell cam example with these combined functions of modified trapezoid and modified sine acceleration.

Example 8.4
Senior Cam Design—Combined Functions—Better Cams.

Problem: Consider the same cam design CEP specification as in Examples 8-1 to 8-3:
Figure 8-22 shows the displacement curves for these three earn programs. (Open the diskfile E08-04.cam in program DYNACAM also.) Note how little difference there is between the displacement curves despite the large differences in their acceleration waveforms in Figure 8-18. This is evidence of the smoothing effect of the integration process. Differentiating any two functions will exaggerate their differences. Integration tends to mask their differences. It is nearly impossible to recognize these very differently behaving earn functions by looking only at their displacement curves. This is further evidence of the folly of our earlier naive approach to earn design which dealt exclusively with the displacement function. The earn designer must be concerned with the higher derivatives of displacement. The displacement function is primarily of value to the manufacturer of the earn who needs its coordinate information in order to cut the earn.

FALL FUNCTIONS We have used only the rise portion of the earn for these examples. The fall is handled similarly. The rise functions presented here are applicable to the fall with slight modification. To convert rise equations to fall equations, it is only necessary to subtract the rise displacement function \( s \) from the maximum lift \( h \) and to negate the higher derivatives, \( v, a, \) and \( j \).

<table>
<thead>
<tr>
<th>dwell</th>
<th>at zero displacement for 90 degrees (low dwell)</th>
</tr>
</thead>
<tbody>
<tr>
<td>rise</td>
<td>1 in (25 mm) in 90 degrees</td>
</tr>
<tr>
<td>dwell</td>
<td>at 1 in (25 mm) for 90 degrees (high dwell)</td>
</tr>
<tr>
<td>fall</td>
<td>1 in (25 mm) in 90 degrees.</td>
</tr>
<tr>
<td>cam ( \theta )</td>
<td>( 2\pi ) rad/sec = 1 rev/sec.</td>
</tr>
</tbody>
</table>

Solution:

1. The modified trapezoidal function is an acceptable one for this double-dwell earn specification. Its derivatives are continuous through the acceleration function as shown in Figures 8-16, 8-18, and 8-20 (pp. 367-371). The peak acceleration is 78.1 in/sec\(^2\) (1.98 m/sec\(^2\)).

2. The modified trapezoidal jerk curve in Figures 8-16 and 8-19 (p. 371) is discontinuous at its boundaries but has finite magnitude of 3925 in/sec\(^3\) (100 m/sec\(^3\)), and this is acceptable.

3. The modified trapezoidal velocity in Figures 8-16 and 8-20 is smooth and matches the zeros of the dwell at each end. Its peak magnitude is 8 in/sec (0.2 m/sec).

4. The advantage of this modified trapezoidal function is that it has smaller peak acceleration than the cycloidal but its peak velocity is identical to that of the cycloidal.

5. The modified sinusoid function is also an acceptable one for this double-dwell earn specification. Its derivatives are also continuous through the acceleration function as shown in Figures 8-18 and 8-20. Its peak acceleration is 88.3 in/sec\(^2\) (2.24 m/sec\(^2\)).

6. The modified sine jerk curve in Figure 8-19 is discontinuous at its boundaries but is of finite magnitude and is larger in magnitude at 4439 in/sec\(^3\) (113 m/sec\(^3\)) but smoother than that of the modified trapezoidal.

7. The modified sine velocity (Figure 8-20) is smooth, matches the zeros of the dwell at each end, and is lower in peak magnitude than either the cycloid or modified trapezoidal at 7 in/sec (0.178 m/sec). This is an advantage for high-mass follower systems as it reduces the kinetic energy. This, coupled with a peak acceleration lower than the cycloidal (but higher than the modified trapezoidal), is its chief advantage.
SUMMARY This section has attempted to present an approach to the selection of appropriate double-dwell cam functions, using the common rise-dwell-fall-dwell cam as the example, and to point out some of the pitfalls awaiting the cam designer. The particular functions described are only a few of the ones that have been developed for this double-dwell case over many years, by many designers, but they are probably the most used and most popular among cam designers. Most of them are also included in program DYNACAM. There are many trade-offs to be considered in selecting a cam program for any application, some of which have already been mentioned, such as function continuity, peak values of velocity and acceleration, and smoothness of jerk. There are many other trade-offs still to be discussed in later sections of this chapter, involving the sizing and the manufacturability of the cam.

8.4 SINGLE-DWELL CAM DESIGN-CHOOSING $SVAJ$ FUNCTIONS

Many applications in machinery require a single-dwell cam program, rise-fall-dwell (RFD). Perhaps a single-dwell cam is needed to lift and lower a roller which carries a moving paper web on a production machine that makes envelopes. This cam's follower lifts the paper up to one critical extreme position at the right time to contact a roller which applies a layer of glue to the envelope flap. Without dwelling in the up position, it immediately retracts the web back to the starting (zero) position and holds it in this other critical extreme position (low dwell) while the rest of the envelope passes by. It repeats the cycle for the next envelope as it comes by. Another common example of a single-
dwell application is the cam which opens the valves in your automobile engine. This lifts
the valve open on the rise, immediately closes it on the fall, and then keeps the valve
closed in a dwell while the compression and combustion take place.

If we attempt to use the same type of cam programs as were defined for the double-
dwell case for a single-dwell application, we will achieve a solution which may work but
is not optimal. We will nevertheless do so here as an example in order to point out the
problems that result. Then we will redesign the cam to eliminate those problems.

EXAMPLE 8-5

Using Cycloidal Motion for Single-Dwell.

Problem: Consider the following single-dwell cam specification:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>rise</td>
<td>1 in (25 mm) in 90 degrees.</td>
</tr>
<tr>
<td>fall</td>
<td>1 in (25 mm) in 90 degrees.</td>
</tr>
<tr>
<td>dwell</td>
<td>at zero displacement for 180 degrees (low dwell).</td>
</tr>
<tr>
<td>cam w0</td>
<td>15 rad/sec.</td>
</tr>
</tbody>
</table>

Solution:

1 Figure 8-23 shows a cycloidal displacement rise and separate cycloidal displacement fall applied to this single-dwell example. Note that the displacement (s) diagram looks acceptable in that it moves the follower from the low to the high position and back in the required intervals.

2 The velocity (v) also looks acceptable in shape in that it takes the follower from zero velocity at the low dwell to a peak value of 19.1 in/sec (0.49 m/sec) to zero again at the maximum displacement, where the glue is applied.

3 Figure 8-23 shows the acceleration function for this solution. Its maximum absolute value is about 573 in/sec².

4 The problem is that this acceleration curve has an unnecessary return to zero at the end of the rise. It is unnecessary because the acceleration during the first part of the fall is also negative. It would be better to keep it in the negative region at the end of the rise.

5 This unnecessary oscillation to zero in the acceleration causes the jerk to have more abrupt changes and discontinuities. The only real justification for taking the acceleration to zero is the need to change its sign (as is the case halfway through the rise or fall) or to match an adjacent segment which has zero acceleration.

The reader may input the file E08-08.cam to program DYNACAM to investigate this example in more detail.

For the single-dwell case we would like a function for the rise which does not return its acceleration to zero at the end of the interval. The function for the fall should begin with the same nonzero acceleration value as ended the rise and then be zero at its terminus to match the dwell. One function which meets those criteria is the double harmonic which gets its name from its two cosine terms, one of which is a half-period harmonic and the other a full-period wave. The equations for the double harmonic functions are:
Cycloidal motion (or any double-dwell program) is a poor choice for the single-dwell case.

for the rise:

\[
\begin{align*}
s &= \frac{h}{2} \left[ \frac{1 - \cos \left( \frac{\theta}{\beta} \right)}{4} \right] \\
v &= \frac{\pi}{\beta} \frac{h}{2} \sin \left( \frac{\theta}{\beta} \right) \\
a &= \frac{\pi^2}{\beta^2} \frac{h}{2} \left[ \cos \left( \frac{\theta}{\beta} \right) - \cos \left( \frac{2\pi}{\beta} \right) \right] \\
j &= -\frac{\pi^2}{\beta^3} \frac{h}{2} \left[ \sin \left( \frac{\theta}{\beta} \right) - 2 \sin \left( \frac{2\pi}{\beta} \right) \right] 
\end{align*}
\]

(8.15a)

for the fall:

\[
\begin{align*}
s &= \frac{h}{2} \left[ \frac{1 + \cos \left( \frac{\theta}{\beta} \right)}{4} \right] \\
v &= -\frac{\pi}{\beta} \frac{h}{2} \sin \left( \frac{\theta}{\beta} \right) + \frac{1}{2} \sin \left( \frac{2\pi}{\beta} \right) \\
a &= -\frac{\pi^2}{\beta^2} \frac{h}{2} \left[ \cos \left( \frac{\theta}{\beta} \right) + \cos \left( \frac{2\pi}{\beta} \right) \right] \\
j &= \frac{\pi^2}{\beta^3} \frac{h}{2} \left[ \sin \left( \frac{\theta}{\beta} \right) + 2 \sin \left( \frac{2\pi}{\beta} \right) \right] 
\end{align*}
\]

(8.15b)
Note that these double harmonic functions should **never** be used for the double-dwell case because their acceleration is nonzero at one end of the interval.

---

**EXAMPLE 8-6**

Double Harmonic Motion for Single-Dwell.

**Problem:** Consider the same single-dwell cam specification as in example 8-5:

- **rise:** 1 in (25 mm) in 90 degrees
- **fall:** 1 in (25 mm) in 90 degrees
- **dwell:** at zero displacement for 180 degrees (low dwell)
- **cam o:** 15 rad/sec

**Solution:**

1. Figure 8-24 shows a double harmonic rise and a double harmonic fall. The peak velocity is 19.5 in/sec (0.5 m/sec) which is similar to that of the cycloidal solution of Example 8-5.

2. Note that the acceleration of this double harmonic function does not return to zero at the end of the rise. This makes it more suitable for a single-dwell case in that respect.

3. The double harmonic jerk function peaks at 36 931 in/sec³ (938 m/sec³) and is quite smooth compared to the cycloidal solution.

4. Unfortunately, the peak negative acceleration is 900 in/sec², nearly twice that of the cycloidal solution. This is a smoother function but will develop higher dynamic forces. Open the diskfile E08-06.cam in program DYNA CAM to see this example in more detail.

---

**FIGURE 8-24**

Double harmonic motion can be used for the single-dwell case if rise and fall durations are equal.
Neither of the solutions in Examples 8-5 and 8-6 is optimal. We will revisit this
example problem after introducing polynomial functions and redesign it yet again to im-
prove both its smoothness and reduce its peak acceleration.

**Summary** This section has presented an approach to the selection of appropriate
single-dwell cam functions and pointed out some of their limitations. The particular
functions described (cycloidal and double harmonic) are only two of those that have been
developed for this single-dwell case. We will visit this single-dwell problem again in the
next section and develop a superior solution using other techniques.

### 8.5 POLYNOMIAL FUNCTIONS

The class of polynomial functions is one of the more versatile types that can be used for
cam design. They are not limited to single- or double-dwell applications and can be tai-
lored to many design specifications. The general form of a polynomial function is:

$$s = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + \cdots + C_n x^n$$  (8.16)

where $s$ is the follower displacement, $x$ is the independent variable, which in our case
will be replaced by either $\theta/\beta$ or time $t$. The constant coefficients $C_n$ are the unknowns
to be determined in our development of the particular polynomial equation to suit a de-
sign specification. The degree of a polynomial is defined as the highest power present
in any term. Note that a polynomial of degree $n$ will have $n + 1$ terms because there is
an $x^0$ or constant term with coefficient $C_0$, as well as coefficients through and including $C_n$.

We structure a polynomial cam design problem by deciding how many boundary
conditions (BCs) we want to specify on the $s v a j$ diagrams. The number of BCs then
determines the degree of the resulting polynomial. We can write an independent equa-
tion for each BC by substituting it into equation 8.16 or one of its derivatives. We will
then have a system of linear equations which can be solved for the unknown coefficients
$C_0, \ldots, C_n$. If $k$ represents the number of chosen boundary conditions, there will be $k$
equations in $k$ unknowns $C_0, \ldots, C_n$ and the degree of the polynomial will be $n = k - 1$.
The order of the $n$-degree polynomial is equal to the number of terms, $k$.

### Double-Dwell Applications of Polynomials

**The 3-4-5 Polynomial** Let us return to the double-dwell problem of Section 8.3
(p. 353) and solve it with polynomial functions. Many different polynomial solutions
are possible. We will start with the simplest one possible for the double-dwell case.

#### EXAMPLE 8-7

The 3-4-5 Polynomial for the Double-Dwell Case.

**Problem:** Consider the same cam design CEP specification as in Examples 8-1 to 8-4:

- dwell at zero displacement for 90 degrees (low dwell)
- rise 1 in (25 mm) in 90 degrees
- dwell at 1 in (25 mm) for 90 degrees (high dwell)
- fall 1 in (25 mm) in 90 degrees
- cam $\omega$ $2\pi$ rad/sec = 1 rev/sec
Solution:

1. To satisfy the fundamental law of cam design, the values of the rise (and fall) functions at their boundaries with the dwell must match with no discontinuities in, at a minimum, $s, v, a, t$.

2. Figure 8-25 shows the axes for the $s, v, a, t$ diagrams on which the known data have been drawn. The dwells are the only fully defined segments at this stage. The requirement for continuity through the acceleration defines a minimum of six boundary conditions for the rise segment and six more for the fall in this problem. They are shown as filled circles on the plots. For generality, we will let the specified total rise be represented by the variable $h$. The minimum set of required BCs for this example is then:

   for the rise:
   
   when $\theta = 0$; then $s = 0, \ v = 0, \ a = 0$

   when $\theta = \beta_1$; then $s = h, \ v = 0, \ a = 0$  \hspace{1cm} (8.17a)

   for the fall:

   when $\theta = 0$; then $s = h, \ v = 0, \ a = 0$

   when $\theta = \beta_2$; then $s = 0, \ v = 0, \ a = 0$  \hspace{1cm} (8.17b)
We will use the rise for an example solution. (The fall is a similar derivation.) We have six BCs on the rise. This requires six terms in the equation. The highest term will be fifth degree. We will use the normalized angle $\theta/\beta$ as our independent variable, as before. Because our boundary conditions involve velocity and acceleration as well as displacement, we need to differentiate equation 8.16 versus $\theta$ to obtain expressions into which we can substitute those BCs. Rewriting equation 8.16 to fit these constraints and differentiating twice, we get:

\[ s = C_0 + C_1 \left( \frac{\theta}{\beta} \right) + C_2 \left( \frac{\theta}{\beta} \right)^2 + C_3 \left( \frac{\theta}{\beta} \right)^3 + C_4 \left( \frac{\theta}{\beta} \right)^4 + C_5 \left( \frac{\theta}{\beta} \right)^5 \]  
(8.18a)

\[ v = \frac{1}{\beta} \left[ C_1 + 2C_2 \left( \frac{\theta}{\beta} \right) + 3C_3 \left( \frac{\theta}{\beta} \right)^2 + 4C_4 \left( \frac{\theta}{\beta} \right)^3 + 5C_5 \left( \frac{\theta}{\beta} \right)^4 \right] \]
(8.18b)

\[ a = \frac{1}{\beta^2} \left[ 2C_2 + 6C_3 \left( \frac{\theta}{\beta} \right) + 12C_4 \left( \frac{\theta}{\beta} \right)^2 + 20C_5 \left( \frac{\theta}{\beta} \right)^3 \right] \]
(8.18c)

4 Substitute the boundary conditions $\theta = 0$, $s = 0$ into equation 8.18a:

\[ 0 = C_0 + 0 + 0 + \ldots \]
(8.19a)

\[ C_0 = 0 \]

5 Substitute $\theta = 0$, $v = 0$ into equation 8.18b:

\[ 0 = \frac{1}{\beta} [C_1 + 0 + 0 + \ldots] \]
(8.19b)

\[ C_1 = 0 \]

6 Substitute $\theta = 0$, $a = 0$ into equation 8.18c:

\[ 0 = \frac{1}{\beta^2} [C_2 + 0 + 0 + \ldots] \]
(8.19c)

\[ C_2 = 0 \]

7 Substitute $\theta = \beta$, $s = h$ into equation 8.18a:

\[ h = C_3 + C_4 + C_5 \]
(8.19d)

8 Substitute $\theta = \beta$, $v = 0$ into equation 8.18b:

\[ 0 = \frac{1}{\beta} [3C_3 + 4C_4 + 5C_5] \]
(8.19e)

9 Substitute $\theta = \beta$, $a = 0$ into equation 8.18c:

\[ 0 = \frac{1}{\beta^2} [6C_3 + 12C_4 + 20C_5] \]
(8.19f)
Three of our unknowns are found to be zero, leaving three unknowns to be solved for, $C_3$, $C_4$, $C_5$. Equations 8.19d, e, and f can be solved simultaneously to get:

$$C_3 = 10h; \quad C_4 = -15h; \quad C_5 = 6h$$  \hspace{1cm} (8.19g)

The equation for this cam design’s displacement is then:

$$s = h \left[ 10 \left( \frac{\theta}{\beta} \right)^3 - 15 \left( \frac{\theta}{\beta} \right)^4 + 6 \left( \frac{\theta}{\beta} \right)^5 \right]$$  \hspace{1cm} (8.20)

The expressions for velocity and acceleration can be obtained by substituting the values of $C_3$, $C_4$, and $C_5$ into equations 8.18b and c. This function is referred to as the 3-4-5 polynomial, after its exponents. Open the file E08-07.cam in program DYNACAM to investigate this example in more detail.

Figure 8-26 shows the resulting $s$ vs $j$ diagrams for a 3-4-5 polynomial rise function from program DYNACAM. Note that the acceleration is continuous but the jerk is not, because we did not place any constraints on the boundary values of the jerk function. It is also interesting to note that the acceleration waveform looks very similar to the sinusoidal acceleration of the cycloidal function in Figure 8-12 (p. 360). Figure 8-18 (p. 370) shows the relative peak accelerations of this 3-4-5 polynomial compared to four other functions with the same $h$ and $\beta$. Table 8-2 (p. 370) lists factors for the maximum velocity, acceleration and jerk of these functions.

**THE 4-5-6-7 POLYNOMIAL** We left the jerk unconstrained in the previous example. We will now redesign the cam for the same specifications but will also constrain the jerk function to be zero at both ends of the rise. It will then match the dwells in the jerk

---

**FIGURE 8-26**

3-4-5 polynomial rise. Its acceleration is very similar to the sinusoid of cycloidal motion.
Any matrix solving calculator, equation solver such as Matlab, Mathcad, or TK Solver, or programs MATRIX and DYNACAM (supplied with this text) will do the simultaneous equation solution for you. Programs MATRIX and DYNACAM are discussed in Appendix A. You need only to supply the desired boundary conditions to DYNACAM and the coefficients will be computed. The reader is encouraged to do so and examine the example problems presented here with the DYNACAM program.

This is known as the 4-5-6-7 polynomial, after its exponents. Figure 8-27 shows the s v a j diagrams for this function. Compare them to the 3-4-5 polynomial functions shown in Figure 8-26 (p. 381). Note that the acceleration of the 4-5-6-7 starts off slowly, with zero slope (as we demanded with our zero jerk BC), and as a result goes to a larger peak value of acceleration in order to replace the missing area in the leading edge.

This 4-5-6-7 polynomial function has the advantage of smoother jerk for better vibration control, compared to the 3-4-5 polynomial, the cycloidal, and all other functions so far discussed (except the double harmonic), but it pays a stiff price in the form of significantly higher acceleration than all those functions. See also Table 8-2 (p. 370).

Single-Dwell Applications of Polynomials

Let us return to the single-dwell example used in the previous section and attempt to solve it with a polynomial cam function. Restating the original problem for reference:
To solve this with a polynomial we must decide on a suitable set of boundary conditions. We must also first decide how many segments to divide the cam cycle into. The problem statement seems to imply three segments, a rise, a fall, and a dwell. We could use those three segments to create the functions as we did in the two previous examples, but a better approach is to use only two segments, one for the rise-fall combined and one for the dwell. As a general rule we would like to minimize the number of segments in our polynomial cam functions. Any dwell requires its own segment. So, the minimum number possible in this case is two segments.

Another rule of thumb is that we would like to minimize the number of boundary conditions specified, because the degree of the polynomial is tied to the number of BCs. As the degree of the function increases, so will the number of its inflection points and its number of minima and maxima. The polynomial derivation process will guarantee that the function will pass through all specified BCs but says nothing about the function’s behavior between the BCs. A high-degree function may have undesirable oscillations between its BCs.

With these assumptions we can select a set of boundary conditions for a trial solution. First we will restate the problem to reflect our two-segment configuration.

EXAMPLE 8-8

Designing a Polynomial for the Single-Dwell Case.

Problem: Redefine the CEP specification from Examples 8-5 and 8-6.

<table>
<thead>
<tr>
<th>Rise</th>
<th>Fall</th>
<th>Dwell</th>
<th>Cam 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 in</td>
<td>1 in (25 mm)</td>
<td>at zero displacement for 180° (low dwell)</td>
<td>15 rad/sec</td>
</tr>
<tr>
<td>(25 mm) in 90°</td>
<td>(25 mm) in 90°</td>
<td>for a total of 180°</td>
<td></td>
</tr>
</tbody>
</table>

Solution:

1. Figure 8-28 shows the minimum set of seven BCs for this problem which will give a sixth-degree polynomial. The dwell on either side of the combined rise-fall segment has zero values of s, v, a, and j. The fundamental law of cam design requires that we match these zero values, through the acceleration function, at each end of the rise-fall segment.

2. These then account for six BCs: s, v, a = 0 at each end of the rise-fall segment.

3. We also must specify a value of displacement at the 1-in peak of the rise which occurs at \( \theta = 90° \). This is the seventh BC.

4. Figure 8-28 also shows the coefficients of the displacement polynomial which result from the simultaneous solution of the equations for the chosen BCs. For generality we have substituted the variable \( h \) for the specified 1-in rise. The function turns out to be a 3-4-5-6 polynomial whose equation is:
SUMMARY This section has presented polynomial functions as the most versatile approach of those shown to virtually any cam design problems. It is only since the development and general availability of computers that these functions have become practical to use, as the computation to solve the simultaneous equations is often beyond hand calculation abilities. With the availability of a design aid to solve the equations such as program DYNACAM, polynomials have become a practical and preferable way to solve many cam design problems. Spline functions, of which polynomials are a subset, offer even more flexibility in meeting boundary constraints and other cam performance criteria.[7] Space does not permit a detailed exposition of spline functions as applied to cam systems here. See the references for more information.

**FIGURE 8-28**
Boundary conditions and coefficients for a single-dwell polynomial application

\[
s = \frac{h}{64} \left( \frac{\theta}{\beta} \right)^3 - 192 \left( \frac{\theta}{\beta} \right)^4 + 192 \left( \frac{\theta}{\beta} \right)^5 - 64 \left( \frac{\theta}{\beta} \right)^6 \tag{8.22}
\]

Figure 8-29 shows the $s v a j$ diagrams for this solution with its maximum values noted. Compare these acceleration and $s v a j$ curves to the double harmonic and cycloidal solutions to the same problem in Section 8.4 (Figures 8-23, p. 376, and 8-24, p. 377). Note that this sixth-degree polynomial function is as smooth as the double harmonic functions (Figure 8-24) and does not unnecessarily return the acceleration to zero at the top of the rise as does the cycloidal (Figure 8-23). The polynomial has a peak acceleration of 547 in/sec$^2$, which is less than that of either the cycloidal or double harmonic solution. This 3-4-5-6 polynomial is a superior solution to either of those presented for the same problem in Section 8.4 and is an example of how polynomial functions can be easily tailored to particular design specifications. The reader may open the file E08-08.cam in program DYNACAM to investigate this example in more detail.

SUMMARY This section has presented polynomial functions as the most versatile approach of those shown to virtually any cam design problems. It is only since the development and general availability of computers that these functions have become practical to use, as the computation to solve the simultaneous equations is often beyond hand calculation abilities. With the availability of a design aid to solve the equations such as program DYNACAM, polynomials have become a practical and preferable way to solve many cam design problems. Spline functions, of which polynomials are a subset, offer even more flexibility in meeting boundary constraints and other cam performance criteria.[7] Space does not permit a detailed exposition of spline functions as applied to cam systems here. See the references for more information.
8.6 CRITICAL PATH MOTION (CPM)

Probably the most common application of critical path motion (CPM) specifications in production machinery design is the need for constant velocity motion. There are two general types of automated production machinery in common use, intermittent motion assembly machines and continuous motion assembly machines.

Intermittent motion assembly machines carry the manufactured goods from work station to work station, stopping the workpiece or subassembly at each station while another operation is performed upon it. The throughput speed of this type of automated production machine is typically limited by the dynamic forces which are due to accelerations and decelerations of the mass of the moving parts of the machine and its workpieces. The workpiece motion may be either in a straight line as on a conveyor or in a circle as on a rotary table as shown in Figure 8-21 (p.372).

Continuous motion assembly machines never allow the workpiece to stop and thus are capable of higher throughput speeds. All operations are performed on a moving target. Any tools which operate on the product have to "chase" the moving assembly line to do their job. Since the assembly line (often a conveyor belt or chain, or a rotary table) is moving at some constant velocity, there is a need for mechanisms to provide constant velocity motion, matched exactly to the conveyor, in order to carry the tools alongside for a long enough time to do their job. These cam driven "chaser" mechanisms must then return the tool quickly to its start position in time to meet the next part or subassembly on the conveyor (quick-return). There is a motivation in manufacturing to convert from intermittent motion machines to continuous motion in order to increase production rates. Thus there is considerable demand for this type of constant velocity mechanism. The earn-follower system is well suited to this problem, and the polynomial earn function is particularly adaptable to the task.
Polynomials Used for Critical Path Motion

EXAMPLE 8-9

Designing a Polynomial for Constant Velocity Critical Path Motion.

Problem: Consider the following statement of a critical path motion (CPM) problem:

- **Accelerate**: the follower from zero to 10 in/sec
- **Maintain**: a constant velocity of 10 in/sec for 0.5 sec
- **Decelerate**: the follower to zero velocity
- **Return**: the follower to start position
- **Cycle time**: exactly 1 sec

Solution:

1. This unstructured problem statement is typical of real design problems as was discussed in Chapter 1. No information is given as to the means to be used to accelerate or decelerate the follower or even as to the portions of the available time to be used for those tasks. A little reflection will cause the engineer to recognize that the specification on total cycle time in effect defines the camshaft velocity to be its reciprocal or **one revolution per second**. Converted to appropriate units, this is an angular velocity of $2\pi$ rad/sec.

2. The constant velocity portion uses half of the total period of 1 sec in this example. The designer must next decide how much of the remaining 0.5 sec to devote to each other phase of the required motion.

3. The problem statement seems to imply that four segments are needed. Note that the designer has to somewhat arbitrarily select the lengths of the individual segments (except the constant velocity one). Some iteration may be required to optimize the result. Program DYNAMCAM makes the iteration process quick and easy, however.

4. Assuming four segments, the timing diagram in Figure 8-30 shows an acceleration phase, a constant velocity phase, a deceleration phase, and a return phase, labeled as segments 1 through 4.

![Motion in inches vs. time](image)

**FIGURE 8-30**

Constant velocity cam timing diagram
3. The segment angles (β's) are assumed, for a first approximation, to be 30° for segment 1, 180° for segment 2, 30° for segment 3, and 120° for segment 4 as shown in Figure 8.31. These angles may need to be adjusted in later iterations, except for segment 2 which is rigidly constrained in the specifications.

6. Figure 8.31 shows a tentative $s v a f$ diagram. The solid circles indicate a set of boundary conditions which will constrain the continuous function to these specifications. These are for segment 1:

   when $\theta = 0^\circ$:  
   \begin{align*}
   s &= 0, \quad v = 0, \quad a = 0
   \end{align*}

   \begin{itemize}
   \item none, $v = 10$, $a = 0$
   \end{itemize}

7. Note that the displacement at $\theta = 30^\circ$ is left unspecified. The resulting polynomial function will provide us with the values of displacement at that point, which can then be used as a

---

**FIGURE 8.31**

A possible set of boundary conditions for the four-segment constant velocity solution.
boundary condition for the next segment, in order to make the overall functions continuous as required. The acceleration at $\theta = 30^\circ$ must be zero in order to match that of the constant velocity segment 2. The acceleration at $\theta = 0$ is left unspecified. The resulting value will be used later to match the end of the last segment's acceleration.

Putting these four BCs for segment 1 into program DYNACAM yields a cubic function whose $s$ vs $\theta$ plots are shown in Figure 8-32. Its equation is:

$$s = 0.83376\left(\frac{\theta}{\beta}\right)^2 - 0.27792\left(\frac{\theta}{\beta}\right)^3$$  \hspace{1cm} (8.23a)

The maximum displacement occurs at $\theta = 30^\circ$. This will be used as one BC for segment 2. The entire set for segment 2 is:

\begin{align*}
\text{when } \theta &= 30^\circ; \quad s = 0.556, \quad v = 10 \\
\text{when } \theta &= 210^\circ; \quad \text{none, none}
\end{align*}

Note that in the derivations and in the DYNACAM program each segment's local angles run from zero to the $\beta$ for that segment. Thus, segment 2's local angles are $0^\circ$ to $180^\circ$, which correspond to $30^\circ$ to $210^\circ$ globally in this example. We have left the displacement, velocity, and acceleration at the end of segment 2 unspecified. They will be determined by the computation.

Since this is a constant velocity segment, its integral, the displacement function, must be a polynomial of degree one, i.e., a straight line. If we specify more than two BCs we will get a function of higher degree than one which will pass through the specified endpoints but may also oscillate between them and deviate from the desired constant velocity. Thus we can only provide two BCs, a slope and an intercept, as defined in equation 8.2 (p. 355). But, we must

---

**Figure 8-32**

Segment one for the four-segment solution to the constant velocity problem.
provide at least one displacement boundary condition in order to compute the coefficient $C_0$ from equation 8.16 (p. 378). Specifying the two BCs at only one end of the interval is perfectly acceptable. The equation for segment 2 is:

$$s = 5 \left( \frac{\theta}{\beta} \right) + 0.556$$  \hspace{1cm} (8.23b)

11 Figure 8-33 shows the displacement and velocity plots of segment 2. The acceleration and jerk are both zero. The resulting displacement at $\theta = 210^\circ$ is 5.556.

12 The displacement at the end of segment 2 is now known from its equation. The four boundary conditions for segment 3 are then:

when $\theta = 210^\circ$: $s = 5.556$, $v = 10$, $a = 0$

when $\theta = 240^\circ$: none, $v = 0$, none

\hspace{1cm} (c)

13 This generates a cubic displacement function as shown in Figure 8-34. Its equation is:

$$s = -0.27792 \left( \frac{\theta}{\beta} \right)^3 + 0.83276 \left( \frac{\theta}{\beta} \right) + 5.556$$  \hspace{1cm} (8.23c)

14 The boundary conditions for the last segment 4 are now defined, as they must match those of the end of segment 3 and the beginning of segment 1. The displacement at the end of segment 3 is found from the computation in DYNACAM to be $s = 6.112$ at $\theta = 240^\circ$ and the acceleration at that point is $-239.9$. We left the acceleration at the beginning of segment 1 unspecified. From the second derivative of the equation for displacement in that segment we find that the acceleration is 239.9 at $\theta = 0^\circ$. The BCs for segment 4 are then:

![Figure 8-33](image)

**FIGURE 8-33**
Segment two for the four-segment solution to the constant velocity problem
The reader may open the file E08-09.cam in program DYNACAM to investigate this example in more detail.

While this design is acceptable, it can be improved. One useful strategy in designing polynomial cams is to minimize the number of segments, provided that this does not result in functions of such high degree that they misbehave between boundary conditions. Another strategy is to always start with the segment for which you have the most information. In this example, the constant velocity portion is the most constrained and must be a separate segment, just as a dwell must be a separate segment. The rest of the cam motion exists only to return the follower to the constant velocity segment for the next cycle. If we start by designing the constant velocity segment, it may be possible to complete the cam with only one additional segment. We will now redesign this cam, to the same specifications but with only two segments as shown in Figure 8-35.

\[ s = -9.9894 \left( \frac{\theta}{\beta} \right)^5 + 24.9738 \left( \frac{\theta}{\beta} \right)^4 - 7.7548 \left( \frac{\theta}{\beta} \right)^3 - 13.3413 \left( \frac{\theta}{\beta} \right)^2 + 6.112 \]  

Figure 8-34 shows the \( s, v, a, j \) plots for the complete cam. It obeys the fundamental law of cam design because the piecewise functions are continuous through the acceleration. The maximum value of acceleration is 257 \( \text{in/sec}^2 \). The maximum negative velocity is \(-29.4 \text{ in/sec}\). We now have four piecewise and continuous functions, equations 8.23, which will meet the performance specifications for this problem.
FIGURE 8-35
Boundary conditions for the two-segment constant velocity solution

EXAMPLE 8-10
Designing an Optimum Polynomial for Constant Velocity Critical Path Motion.

Problem: Redefine the problem statement of Example 8-9 to have only two segments.

Maintain a constant velocity of 10 in/sec for 0.5 sec
Deaccelerate and accelerate follower to constant velocity
Cycle time exactly 1 sec

Solution:

1. The BCs for the first, constant velocity, segment will be similar to our previous solution except for the global values of its angles and the fact that we will start at zero displacement. They are:
   
   when $\theta = 0^\circ$; $s = 0$, $v = 10$
   
   when $\theta = 180^\circ$; none, none

2. The displacement and velocity plots for this segment are identical to those in Figure 8-33 (p. 389) except that the displacement starts at zero. The equation for segment 1 is:

$$s = \frac{\theta}{\beta}$$

(8.24a)
3 The program calculates the displacement at the end of segment 1 to be 5.00 in. This defines that BC for segment 2. The set of BCs for segment 2 is then:

\[
\begin{align*}
\text{when } \theta &= 180^\circ; \quad & s &= 5.00, & v &= 10, & a &= 0 \\
\text{when } \theta &= 360^\circ; \quad & s &= 0, & v &= 10, & a &= 0
\end{align*}
\]

The equation for segment 2 is:

\[
s = -60 \left( \frac{\theta}{\beta} \right)^5 + 30 \left( \frac{\theta}{\beta} \right)^4 - 100 \left( \frac{\theta}{\beta} \right)^3 + 5 \left( \frac{\theta}{\beta} \right)^2 + 5 \quad (8.24b)
\]

4 The s vs a vs j diagrams for this design are shown in Figure 8-36. Note that they are much smoother than the four-segment design. The maximum acceleration in this example is now 230 in/sec², and the maximum negative velocity is −27.5 in/sec. These are both less than in the previous design of Example 8-9.

5 The fact that our displacement in this design contains negative values, as shown in the s diagram of Figure 8-36 is of no concern. This is due to our starting with the beginning of the constant velocity portion as zero displacement. The follower has to go to a negative position in order to have distance to accelerate up to speed again. We will simply shift the displacement coordinates by that negative amount to make the cam. To do this, simply calculate the displacement coordinates for the cam. Note the value of the largest negative displacement. Add this value to the displacement boundary conditions for all segments and recalculate the cam functions with DYNACAM. (Do not change the BCs for the higher derivatives.) The finished cam’s displacement profile will be shifted up such that its minimum value will now be zero.

![Figure 8-36](image)

**Figure 8-36**

Two-segment solution to the constant velocity problem showing maximum values
So, not only do we now have a smoother cam but the dynamic forces and stored kinetic energy are both lower. Note that we did not have to make any assumptions about the portions of the available nonconstant velocity time to be devoted to speeding up or slowing down. This all happened automatically from our choice of only two segments and the specification of the minimum set of necessary boundary conditions. This is clearly a superior design to the previous attempt and is in fact an optimal polynomial solution to the given specifications. The reader is encouraged to read the file E08-10.cam into program DYNACAM to investigate this example in more detail.

**Half-Period Harmonic Family Functions**

The full-rise cycloidal and the full-rise modified sine functions are generally suited only to the double-dwell cases as they have zero acceleration at each end. However, pieces of these functions can be used to match other functions such as constant velocity segments in a similar fashion to that used to piece together the modified sine from two harmonics of different frequency. The full-rise functions mentioned above (except simple harmonic) contain one complete period in their velocity and acceleration. The simple harmonic (Figure 8-9, p. 360) does not have zero acceleration at its ends, but the modified sine’s and cycloid’s accelerations (Figure 8-12, p. 358) do start and end at zero. In order to match a nonzero velocity as in the example above, we could use half of any of these harmonic family functions and design them to match the desired constant velocity of the adjacent segment.

As an example of this approach, Figure 8-37 shows the $s$, $v$, $a$, $j$ functions for a half-cycloidal rise function #1 which has zero velocity at the beginning of the interval and non-zero velocity at the end. Note that its displacement starts at zero and ends at some positive value, but its acceleration is zero at both extremes. This makes it possible to match this function to a constant velocity segment and match both the desired velocity and its zero acceleration at the boundary. The total displacement required of the half-cycloid will “come out in the wash” when the required boundary conditions of velocity and duration are applied to the particular case. The equations for this half-cycloid #1 are:

---

**FIGURE 8-37**

Half-cycloidal functions for use on a rise segment
For a fall instead of a rise, subtract the rise displacement expressions from the total rise $L$ and negate all the higher derivatives.

To fit these functions to a particular constant velocity situation, solve either equation 8.25b or 8.26b (depending on which function is desired) for the value of $L$ which results from the specification of the known constant velocity $v$ to be matched at $\theta = \pi$ or $\theta = 0$. You will have to choose a value of $\pi$ for the interval of this half-cycloid which is appropriate to the problem. In our example above, the value of $\pi = 30^o$ used for the first segment of the four-piece polynomial could be tried as a first iteration. Once $L$ and $\pi$ are known, all the functions are defined.

The same approach can be taken with the modified sine and the simple harmonic functions. Either half of their full-rise functions can be sized to match with a constant velocity segment. The half-modified sine function mated with a constant velocity segment has the advantage of low peak velocity, useful with large inertia loads. When matched to a constant velocity, the half simple harmonic has the same disadvantage of infinite jerk as its full-rise counterpart does when matched to a dwell, so it is not recommended.

We will now solve the previous constant velocity example problem using half-cycloid, constant velocity, and full-fall modified sine functions.
EXAMPLE 8-11

Using Half-Cycloids to Match Constant Velocity Critical Path Motion.

Problem: Consider the same problem statement as Example 8-9.

- Accelerate: the follower from zero to 10 in/sec
- Maintain: a constant velocity of 10 in/sec for 0.5 sec
- Decelerate: the follower to zero velocity
- Return: the follower to start position
- Cycle time: exactly 1 sec

Solution:

1. We must express the specified constant velocity in units of length per rad. The angular velocity is $2\pi$ rad/sec.
   \[
   \omega = 10 \text{ in/sec} \left( \frac{1 \text{ sec}}{2\pi \text{ rad}} \right) = \frac{5}{\pi} \text{ in/} \text{rad}
   \]

2. The constant velocity portion uses half of the total period of 1 sec, or $\pi$ rad, in this example. The designer must decide how much of the remaining 0.5 sec to devote to each other phase of the required motion. The segment angles ($\beta$'s) are assumed, for a first approximation, to be 25° for segment 1, 180° for segment 2, 25° for segment 3, and 130° for segment 4. These angles may need to be adjusted in later iterations to balance and minimize the accelerations (except for segment 2 which is rigidly constrained in the specifications).

3. The segments will consist of:

<table>
<thead>
<tr>
<th>Segment</th>
<th>$\beta$ (deg)</th>
<th>$\beta$ (rad)</th>
<th>Function</th>
<th>Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>0.43633</td>
<td>Half-cycloid #1</td>
<td>Rise</td>
</tr>
<tr>
<td>2</td>
<td>180</td>
<td>3.14159</td>
<td>1° polynomial</td>
<td>Rise</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>0.43633</td>
<td>Half-cycloid #2</td>
<td>Rise</td>
</tr>
<tr>
<td>4</td>
<td>130</td>
<td>2.26893</td>
<td>Modified sine</td>
<td>Fall</td>
</tr>
</tbody>
</table>

4. To determine the total rise $L$ of the half-cycloid needed to match the specified constant velocity, solve equation 8.25b for $L$ at $\theta = \beta$ where it must match the constant velocity $\nu$:
   \[
   L = \frac{\beta \nu}{1 - \cos \left( \frac{\pi \theta}{\beta} \right)} = \frac{0.43633 \left( \frac{4}{\pi} \right)}{1 - \cos \left( \frac{\pi \theta}{\beta} \right)} = 0.34722
   \]

5. Substitute this value of $L$ in equation 8.25a to get the displacement for the first segment:
   \[
   s = 0.34722 \left[ \frac{\theta}{\beta} - \frac{1}{\beta} \sin \left( \frac{\pi \theta}{\beta} \right) \right]
   \]
6 The constant velocity segment is found in the same way as was done in Example 8-9 (p. 386). The initial displacement for segment 2 in this case is the value of \( L \), and the equation for segment 2 is:

\[
s = 5 \left( \frac{\theta}{\beta} \right) + 0.3472
\]

The total lift within this segment is 5 in as before.

7 Segment 3 is a half-cycloid #2. Its coefficient \( L \) is identical to that of segment 1 because we used the same \( \beta \) for both. But, we must offset it by the sum of the displacement of segments 2 and 3 or \( L + 5 \). Program DYNACAM provides for the specification of this offset. The lift for this segment is 0.3472 and the offset is 5.3472. The equation for segment 3 (from equation 8.26a) is then:

\[
s = 0.3472 \left[ \frac{\theta}{\beta} + \frac{1}{\pi} \sin \left( \pi \frac{\theta}{\beta} \right) \right] + 5.3472
\]

8 Segment 4 is a full-period modified sine to return the follower from its maximum displacement of \( L = 0.3472 + 5.0 + 0.3472 = 5.6944 \). See equation 8.14 (p. 369).

9 The complete set of data needed to compute these functions in (or out of) DYNACAM is:

<table>
<thead>
<tr>
<th>Segment</th>
<th>( \beta ) (deg)</th>
<th>( \beta ) (rad)</th>
<th>Function</th>
<th>Start (in)</th>
<th>Motion</th>
<th>Move (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>0.43633</td>
<td>Half-cycloid #1</td>
<td>0</td>
<td>Rise</td>
<td>0.3472</td>
</tr>
<tr>
<td>2</td>
<td>180</td>
<td>3.14159</td>
<td>1st polynomial</td>
<td>0.3472</td>
<td>Rise</td>
<td>5.0000</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>0.43633</td>
<td>Half-cycloid #2</td>
<td>5.3472</td>
<td>Rise</td>
<td>0.3472</td>
</tr>
<tr>
<td>4</td>
<td>130</td>
<td>2.26893</td>
<td>Modified sine</td>
<td>5.6944</td>
<td>Fall</td>
<td>5.6944</td>
</tr>
</tbody>
</table>

10 The resulting \( s \) vs \( \theta \) diagrams are shown in Figure 8-38. The peak acceleration is 241 in/sec\(^2\) and peak velocity is -28 in/sec.

These results are nearly as low as the values from the two-segment polynomial solution in Example 8-10 (p. 391). The factor that makes this an inferior cam design to Example 8-10 is the unnecessary returns to zero in the acceleration waveform. This creates a more "ragged" jerk function which will increase vibration problems. The polynomial approach is superior to the other solutions presented in this case as it often is in cam design. The reader may open the file E08-11.cam in program DYNACAM to investigate this example in more detail.

### 8.7 SIZING THE CAM-PRESSURE ANGLE AND RADIUS OF CURVATURE

Once the \( s \) vs \( \theta \) functions have been defined, the next step is to size the cam. There are two major factors which affect cam size, the **pressure angle** and the **radius of curvature**. Both of these involve either the base circle radius on the cam \( (R_b) \) when using flat-faced followers, or the prime circle radius on the cam \( (R_p) \) when using roller or curved followers.
The base circle's and prime circle's centers are at the center of rotation of the earn. The base circle is defined as the smallest circle which can be drawn tangent to the physical cam surface as shown in Figure 8-39. All radial cams will have a base circle, regardless of the follower type used.

The prime circle is only applicable to cams with roller followers or radiused (mushroom) followers and is measured to the center of the follower. The prime circle is defined as the smallest circle which can be drawn tangent to the locus of the centerline of the follower as shown in Figure 8-39. The locus of the centerline of the follower is called the pitch curve. Cams with roller followers are in fact defined for manufacture with respect to the pitch curve rather than with respect to the earn's physical surface. Cams with flat-faced followers must be defined for manufacture with respect to their physical surface, as there is no pitch curve.

The process of creating the physical earn from the s diagram can be visualized conceptually by imagining the s diagram to be cut out of a flexible material such as rubber. The x axis of the s diagram represents the circumference of a circle, which could be either the base circle, or the prime circle, around which we will "wrap" our "rubber" s diagram. We are free to choose the initial length of our s diagram's axis, though the height of the displacement curve is fixed by the earn displacement function we have chosen. In effect we will choose the base or prime circle radius as a design parameter and stretch the length of the s diagram's axis to fit the circumference of the chosen circle.

**Pressure Angle-Roller Followers**

The pressure angle is defined as shown in Figure 8-40. It is the complement of the transmission angle which was defined for linkages in previous chapters and has a similar meaning with respect to earn-follower operation. By convention, the pressure angle is used for cams, rather than the transmission angle. Force can only be transmitted from
earn to follower or vice versa along the axis of transmission which is perpendicular to the axis of slip, or common tangent.

PRESSURE ANGLE The pressure angle is the angle between the direction of motion (velocity) of the follower and the direction of the axis of transmission. When \( \phi = 0 \), all the transmitted force goes into motion of the follower and none into slip velocity. When \( \phi \) becomes 90° there will be no motion of the follower. As a rule of thumb, we would like the pressure angle to be between zero and about 30° for translating followers to avoid excessive side load on the sliding follower. If the follower is oscillating on a pivoted arm, a pressure angle up to about 35° is acceptable. Values of \( \phi \) greater than this can increase the follower sliding or pivot friction to undesirable levels and may tend to jam a translating follower in its guides.

ECCENTRICITY Figure 8-41 shows the geometry of a earn and translating roller follower in an arbitrary position. This shows the general case in that the axis of motion of the follower does not intersect the center of the earn. There is an eccentricity \( \varepsilon \) defined as the perpendicular distance between the follower's axis of motion and the center of the cam. Often this eccentricity \( \varepsilon = 0 \), making it an aligned follower, which is the special case.

In the figure, the axis of transmission is extended to intersect effective link 1, which is the ground link. (See Section 8.0 and Figure 8-1, p. 347 for a discussion of effective links in earn systems.) This intersection is instant center \( \varepsilon,4 \) (labeled \( \delta \)) which, by definition, has the same velocity in link 2 (the earn) and in link 4 (the follower). Because link 4 is in pure translation, all points on it have identical velocities \( v_{\text{follower}} \) which are

---

* Dresner points out that this definition is only valid for single-degree-of-freedom systems. For multi-input systems, a more complicated definition and calculation of pressure angle (or transmission angle) is needed. For more information see Dresner, T. L. and K. W. Buffington. (1991). "Definition of Pressure and Transmission Angles Applicable to Multi-Input Mechanisms," *Journal of Mechanical Design*, 113(4), p. 495.
equal to the velocity of $I_{2,4}$ in link 2. We can write an expression for the velocity of $I_{2,4}$
in terms of cam angular velocity and the radius $b$ from cam center to $I_{2,4}$,

$$V_{I_{2,4}} = b\omega = \dot{s}$$  \hspace{1cm} (8.27)

where $s$ is the instantaneous displacement of the follower from the $s$ diagram and $s\dot{}$ is its time derivative in units of length/sec. (Note that capital $S\ V\ A\ J$ denote time-based variables rather than functions of cam angle.)

But

$$\dot{s} = \frac{ds}{dt}$$

and

$$\frac{ds}{dt} = \frac{ds}{d\theta} \frac{d\theta}{dt} = \frac{ds}{d\theta} \dot{\theta} = \omega = v_0$$

so

$$b\omega = v_0$$

then

$$b = v$$  \hspace{1cm} (8.28)
This is an interesting relationship which says that the distance \( b \) to the instant center 12,4 is numerically equal to the velocity of the follower \( v \) in units of length per radian as derived in previous sections. We have reduced this expression to pure geometry, independent of the angular velocity \( \omega \) of the cam.

Note that we can express the distance \( b \) in terms of the prime circle radius \( R_p \) and the eccentricity \( e \), by the construction shown in Figure 8-41. Swing the arc of radius \( R_p \) until it intersects the axis of motion of the follower at point \( D \). This defines the length of line \( d \) from effective link 1 to this intersection. This is constant for any chosen prime circle radius \( R_p \). Points \( A, C \), and \( D, 2, 4 \) form a right triangle whose upper angle is the pressure angle \( \phi \) and whose vertical leg is \( s + d \), where \( s \) is the instantaneous displacement of the follower. From this triangle:

\[
e = (s + d) \tan \phi
\]

and

\[
b = (s + d) \tan \phi + e
\]  

Then from equation 8.28,

\[
v = (s + d) \tan \phi + e
\]  

and from triangle \( CDO_2 \),

\[
d = \sqrt{R_p^2 - e^2}
\]  

Substituting equation 8.29c into equation 8.29b and solving for \( \phi \) gives an expression for pressure angle in terms of displacement \( s \), velocity \( v \), eccentricity \( e \), and the prime circle radius \( R_p \).

\[
\phi = \arctan \frac{v - e}{s + \sqrt{R_p^2 - e^2}}
\]  

The velocity \( v \) in this expression is in units of length/rad, and all other quantities are in compatible length units. We have typically defined \( s \) and \( v \) by this stage of the cam design process and wish to manipulate \( R_p \) and \( e \) to get an acceptable maximum pressure angle \( \phi \). As \( R_p \) is increased, \( \phi \) will be reduced. The only constraints against large values of \( R_p \) are the practical ones of package size and cost. Often there will be some upper limit on the size of the cam-follower package dictated by its surroundings. There will always be a cost constraint and bigger = heavier = more expensive.

**Choosing a Prime Circle Radius**

Both \( R_p \) and \( e \) are within a transcendental expression in equation 8.29d, so they cannot be conveniently solved for directly. The simplest approach is to assume a trial value for \( R_p \) and an initial eccentricity of zero, and use program DYNACAM; your own program; or an equation solver such as Matlab, TKSolver or Mathcad to quickly calculate the values of \( e \) for the entire earn, and then adjust \( R_p \) and repeat the calculation until an acceptable arrangement is found. Figure 8-42 shows the calculated pressure angles for a four-dwell earn. Note the similarity in shape to the velocity functions for the same earn in Figure 8-6 (p. 353), as that term is dominant in equation 8.29d.
If a suitably small cam cannot be obtained with acceptable pressure angle, then eccentricity can be introduced to change the pressure angle. Using eccentricity to control the pressure angle has its limitations. For a positive \( \varepsilon \), a positive value of eccentricity will decrease the pressure angle on the rise but will increase it on the fall. Negative eccentricity does the reverse.

This is of little value with a form-closed (groove or track) cam, as it is driving the follower in both directions. For a force-closed cam with spring return, you can sometimes afford to have a larger pressure angle on the fall than on the rise because the stored energy in the spring is attempting to speed up the camshaft on the fall, whereas the cam is storing that energy in the spring on the rise. The limit of this technique can be the degree of overspeed attained with a larger pressure angle on the fall. The resulting variations in cam angular velocity may be unacceptable.
The most value gained from adding eccentricity to a follower comes in situations where the cam program is asymmetrical and significant differences exist (with no eccentricity) between maximum pressure angles on rise and fall. Introducing eccentricity can balance the pressure angles in this situation and create a smoother running cam.

If adjustments to $R_p$ or $e$ do not yield acceptable pressure angles, the only recourse is to return to an earlier stage in the design process and redefine the problem. Less lift or more time to rise or fall will reduce the causes of the large pressure angle. Design is, after all, an iterative process.

**Overturing Moment-Flat-Faced Follower**

Figure 8-43 shows a translating, flat-faced follower running against a radial cam. The pressure angle can be seen to be zero for all positions of cam and follower. This seems to be giving us something for nothing, which can't be true. As the contact point moves left and right, the point of application of the force between cam and follower moves with it. There is an overturning moment on the follower associated with this off-center force which tends to jam the follower in its guides, just as did too large a pressure angle in the roller follower case. In this case, we would like to keep the cam as small as possible in order to minimize the moment arm of the force. Eccentricity will affect the average value of the moment, but the peak-to-peak variation of the moment about that average is unaffected by eccentricity. Considerations of too-large pressure angle do not limit the size of this cam, but other factors do. The minimum radius of curvature (see below) of the cam surface must be kept large enough to avoid undercutting. This is true regardless of the type of follower used.
Radius of Curvature—Roller Follower

The radius of curvature is a mathematical property of a function. Its value and use is not limited to cams but has great significance in their design. The concept is simple. No matter how complicated a curve’s shape may be, nor how high the degree of the function which describes it, it will have an instantaneous radius of curvature at every point on the curve. These radii of curvature will have instantaneous centers (which may be at infinity), and the radius of curvature of any function is itself a function which can be computed and plotted. For example, the radius of curvature of a straight line is infinity everywhere; that of a circle is a constant value. A parabola has a constantly changing radius of curvature which approaches infinity along the parabola’s asymptotes. A cubic curve will have radii of curvature that are sometimes positive (convex) and sometimes negative (concave). The higher the degree of a function, in general, the more potential variety in its radius of curvature.

\[
\Sigma M = 0
\]

\[F_{\text{cam}} \times d = F_b \times d\]

**FIGURE 8-43**

Overturning moment on a flat-faced follower
Cam contours are usually functions of high degree. When they are wrapped around their base or prime circles, they may have portions which are concave, convex, or flat. Infinitesimally short flats of infinite radius will occur at all inflection points on the cam surface where it changes from concave to convex or vice versa.

The radius of curvature of the finished cam is of concern regardless of the follower type, but the concerns are different for different followers. Figure 8-44 shows an obvious problem with a roller follower whose own (constant) radius of curvature $R_f$ is too large to follow the locally smaller concave (negative) radius $-\rho$ on the cam.

A more subtle problem occurs when the roller follower radius $R_f$ is larger than the smallest positive (convex) local radius $+\rho$ on the cam. This problem is called undercutting and is depicted in Figure 8-45. Recall that for a roller follower cam, the cam contour is actually defined as the locus of the center of the roller follower, or the pitch curve. The machinist is given these $x,y$ coordinate data (on computer tape or disk) and also told the radius of the follower $R_f$. The machinist will then cut the cam with a cutter of the same effective radius as the follower, following the pitch curve coordinates with the center of the cutter.

Figure 8-45a shows the situation in which the follower (cutter) radius $R_f$ is at one point exactly equal to the minimum convex radius of curvature of the cam ($+\rho_{\text{min}}$). The cutter creates a perfect sharp point, or cusp, on the cam surface. This cam will not run very well at speed! Figure 8-45b shows the situation in which the follower (cutter) radius is greater than the minimum convex radius of curvature of the cam. The cutter now undercuts or removes material needed for cam contours in different locations and also creates a sharp point or cusp on the cam surface. This cam no longer has the same displacement function you so carefully designed.

The rule of thumb is to keep the absolute value of the minimum radius of curvature $\rho_{\text{min}}$ of the cam pitch curve preferably at least 2 to 3 times as large as the radius of the roller follower $R_f$.

\[ R_f > |\rho_{\text{min}}| \]

**FIGURE 8-44**

The result of using a roller follower larger than the one for which the cam was designed.
Small positive radius of curvature can cause undercutting.

\[ |\rho_{\text{min}}| \gg R_f \] (8.30)

A derivation for radius of curvature can be found in any calculus text. For our case of a roller follower, we can write the equation for the radius of curvature of the pitch curve of the cam as:

\[ \rho_{\text{pitch}} = \frac{[(R_p + s)^2 + v^2]^{3/2}}{(R_p + s)^2 + 2v^2 - a(R_p + s)} \] (8.31)

In this expression, \( s \), \( v \), and \( a \) are the displacement, velocity, and acceleration of the cam program as defined in a previous section. Their units are length, length/\( \text{rad} \), and length/\( \text{rad}^2 \), respectively. \( R_p \) is the prime circle radius. **Do not confuse** this prime circle radius \( R_p \) with the radius of curvature, \( \rho_{\text{pitch}} \). \( R_p \) is a **constant value** which you choose as a design parameter and \( \rho_{\text{pitch}} \) is the constantly changing radius of curvature which results from your design choices.

Also do not confuse \( R_p \), the prime circle radius with \( R_f \), the radius of the roller follower. See Figure 8-39 (p. 398) for definitions. You can choose the value of \( R_f \) to suit the problem, so you might think that it is simple to satisfy equation 8.30 by just selecting...
a roller follower with a small value of $R_f$. Unfortunately it is more complicated than that, as a small roller follower may not be strong enough to withstand the dynamic forces from the cam. The radius of the pin on which the roller follower pivots is substantially smaller than $R_f$ because of the space needed for roller or ball bearings within the follower. Dynamic forces will be addressed in later chapters where we will revisit this problem.

We can solve equation 8.31 for $\rho_{\text{pitch}}$ since we know $x$, $y$, and $a$ for all values of $\theta$ and can choose a trial $R_p$. If the pressure angle has already been calculated, the $R_p$ found for its acceptable values should be used to calculate $\rho_{\text{pitch}}$ as well. If a suitable follower radius cannot be found which satisfies equation 8.30 for the minimum values of $\rho_{\text{pitch}}$ calculated from equation 8.31, then further iteration will be needed, possibly including a redefinition of the cam specifications.

Program DYNACAM calculates $\rho_{\text{pitch}}$ for all values of $\theta$ for a user supplied prime circle radius $R_p$. Figure 8-46 shows $\rho_{\text{pitch}}$ for the four-dwell cam of Figure 8-6 (p. 353). Note that this cam has both positive and negative radii of curvature. The large values of radius of curvature are truncated at arbitrary levels on the plot as they are heading to infinity at the inflection points between convex and concave portions. Note that the radii of curvature go out to positive infinity and return from negative infinity or vice versa at these inflection points (perhaps after a round trip through the universe?).

Once an acceptable prime circle radius and roller follower radius are determined based on pressure angle and radius of curvature considerations, the cam can be drawn in finished form and subsequently manufactured. Figure 8-47 shows the profile of the four-dwell cam from Figure 8-6. The cam surface contour is swept out by the envelope of follower positions just as the cutter will create the cam in metal. The sidebar shows the parameters for the design, which is an acceptable one. The $p_{\text{min}}$ is 1.7 times $R_f$ and the pressure angles are less than 30°. The contours on the cam surface appear smooth, with no sharp corners. Figure 8-48 shows the same cam with only one change. The radius of follower $R_f$ has been made the same as the minimum radius of curvature, $p_{\text{min}}$. The sharp corners or cusps in several places indicate that undercutting has occurred. This has now become an unacceptable cam, simply because of a roller follower that is too large.

The coordinates for the cam contour, measured to the locus of the center of the roller follower, or the pitch curve as shown in Figure 8-47, are defined by the following expressions, referenced to the center of rotation of the cam. See Figure 8-42 (p. 402) for nomenclature. The subtraction of the cam input angle $\theta$ from $2\pi$ is necessary because the relative motion of the follower versus the cam is opposite to that of the cam versus the follower. In other words, to define the contour of the centerline of the follower’s path around a stationary cam, we must move the follower (and also the cutter to make the cam) in the opposite direction of cam rotation.

\[
x = \cos \lambda \sqrt{(d + s)^2 + \varepsilon^2}
\]

\[
y = \sin \lambda \sqrt{(d + s)^2 + \varepsilon^2}
\]

where:

\[
\lambda = (2\pi - \theta) - \arctan \left( \frac{\varepsilon}{d + s} \right)
\]
Radius of Curvature—Flat-Faced Follower

The situation with a flat-faced follower is different to that of a roller follower. A negative radius of curvature on the cam cannot be accommodated with a flat-faced follower. The flat follower obviously cannot follow a concave cam. Undercutting will occur when the radius of curvature becomes negative if a cam with that condition is made.
Figure 8-49 shows a cam and flat-faced follower in an arbitrary position. The origin of the global XY coordinate system is placed at the cam’s center of rotation, and the X axis is defined parallel to the common tangent, which is the surface of the flat follower. The vector r is attached to the cam, rotates with it, and serves as the reference line to which the cam angle θ is measured from the X axis. The point of contact A is defined by the position vector R_A. The instantaneous center of curvature is at C and the radius of curvature is p. R_b is the radius of the base circle and s is the displacement of the follower for angle θ. The eccentricity is e.

We can define the location of contact point A from two vector loops (in complex notation):

\[ R_A = x + j(R_b + s) \]

and

\[ R_A = ce^{j(θ+α)} + jp \]

so:

\[ ce^{j(θ+α)} + jp = x + j(R_b + s) \]  \hspace{1cm} (8.33a)

Substitute the Euler equivalent (equation 4.4a, p. 155) in equation 8.33a and separate the real and imaginary parts.

real:

\[ c\cos(θ + α) = x \]  \hspace{1cm} (8.33b)

imaginary:

\[ c\sin(θ + α) + p = R_b + s \]  \hspace{1cm} (8.33c)
The center of curvature \( C \) is **stationary** on the cam meaning that the magnitudes of \( c \) and \( p \), and angle \( \alpha \), do not change for small changes in cam angle \( \theta \). (These values are not constant but are at stationary values. Their first derivatives with respect to \( \theta \) are zero, but their higher derivatives are not zero.)

Differentiating equation 8.33a with respect to \( \theta \) then gives:

\[
j c e^{j(\theta + \alpha)} = \frac{dx}{d\theta} + j \frac{dx}{d\theta}
\]  

(8.34)

Substitute the Euler equivalent (equation 4.4a, p. 155) in equation 8.34 and separate the real and imaginary parts.

**real:**

\[-c \sin(\theta + \alpha) = \frac{dx}{d\theta} \]  

(8.35)

**imaginary:**

\[c \cos(\theta + \alpha) = \frac{dx}{d\theta} = v \]  

(8.36)

Inspection of equations 8.33b and 8.36 shows that:
\[ x = v \]  

(8.37)

This is an interesting relationship that says the \( x \) position of the contact point between cam and follower is numerically equal to the velocity of the follower in length/rad. This means that the \( v \) diagram gives a direct measure of the necessary minimum face width of the flat follower.

\[ \text{facewidth} > v_{\text{max}} - v_{\text{min}} \]  

(8.38)

If the velocity function is asymmetric, then a minimum-width follower will have to be asymmetric also, in order not to fall off the cam.

Differentiating equation 8.37 with respect to \( \theta \) gives:

\[ \frac{dx}{d\theta} = \frac{dv}{d\theta} = a \]  

(8.39)

Equations 8.33c and 8.35 can be solved simultaneously and equation 8.39 substituted in the result to yield:

\[ \rho = R_b + s + a \]  

(8.40)

**BASE CIRCLE**  Note that equation 8.40 defines the radius of curvature in terms of the base circle radius and the displacement and acceleration functions from the \( s \ v \ a \) diagrams only. Because \( \rho \) cannot be allowed to become negative with a flat-faced follower, we can formulate a relationship from this equation which will predict the minimum base circle radius \( R_b \) needed to avoid undercutting. The only factor on the right side of equation 8.40 which can be negative is the acceleration, \( a \). We have defined \( s \) to be always positive, as is \( R_b \). Therefore, the worst case for undercutting will occur when \( a \) is at its largest negative value, \( a_{\text{min}} \), whose value we know from the \( a \) diagram. The minimum base circle radius can then be defined as:

\[ R_{b_{\text{min}}} > R_b + s + a_{\text{min}} \]  

(8.41)

Note that the value of \( s \) in this equation is taken at the cam angle \( \theta \) corresponding to that of \( a_{\text{min}} \). Because the value of \( a_{\text{min}} \) is negative and it is also negated in equation 8.41, it dominates the expression. To use this relationship, we must choose some minimum radius of curvature \( R_{b_{\text{min}}} \) for the cam surface as a design parameter. Since the hertzian contact stresses at the contact point are a function of local radius of curvature, this criterion can be used to select \( R_{b_{\text{min}}} \). That topic is beyond the scope of this text and will not be further explored here. See reference 1 for further information on contact stresses.

**CAM CONTOUR**  For a flat-faced follower cam, the coordinates of the physical cam surface must be provided to the machinist as there is no pitch curve to work to. Figure 8-49 (p. 409) shows two orthogonal vectors, \( r \) and \( q \), which define the cartesian coordinates of contact point \( A \) between cam and follower with respect to a rotating axis coordinate system embedded in the cam. Vector \( r \) is the rotating "s" axis of this embedded coordinate system. Angle \( \psi \) defines the position of vector \( R_A \) in this system. Two vector loop equations can be written and equated to define the coordinates of all points on the cam surface as a function of cam angle \( \theta \).
\[ R_A = x + j(R_b + s) \]

and

\[ R_A = re^{j\theta} + qe^{j(\theta + \frac{\pi}{2})} \]

so:

\[ re^{j\theta} + qe^{j(\theta + \frac{\pi}{2})} = x + j(R_b + s) \quad (8.42) \]

Divide both sides by \( e^{j\theta} \):

\[ r + jq = xe^{-j\theta} + j(R_b + s)e^{-j\theta} \quad (8.43) \]

Separate into real and imaginary components and substitute \( v \) for \( x \) from equation 8.37:

real (x component):

\[ r = (R_b + s)\sin\theta + v\cos\theta \quad (8.44a) \]

imaginary (y component):

\[ q = (R_b + s)\cos\theta - v\sin\theta \quad (8.44b) \]

Equations 8.44 can be used to machine the cam for a flat-faced follower. These \( x, y \) components are in the rotating coordinate system that is embedded in the cam.

Note that none of the equations developed above for this case involve the eccentricity, \( e \). It is only a factor in cam size when a roller follower is used. It does not affect the geometry of a flat follower cam.

Figure 8-50 shows the result of trying to use a flat-faced follower on a cam with negative radius of curvature. If the follower could contact the cam at all the points necessary to control the follower to the function in the \( s \) diagram, the cam surface would be as developed by the envelope of straight lines. However, these loci of the follower face are cutting into cam contours which are needed for other cam angles. The line running through the forest of follower loci is the theoretical cam contour needed for this design. The undercutting can be clearly seen as the crescent-shaped missing pieces at four places between the cam contour and the follower loci.

**Summary** The task of sizing a cam is an excellent example of the need for and value of iteration in design. Rapid recalculation of the relevant equations with a tool such as program DYNACAM makes it possible to quickly and painlessly arrive at an acceptable solution while balancing the often conflicting requirements of pressure angle and radius of curvature constraints. In any cam, either the pressure angle or radius of curvature considerations will dictate the minimum size of the cam. Both factors must be checked. The choice of follower type, either roller or flat-faced, makes a big difference in the cam geometry. Cam programs which generate negative radii of curvature are unsuited to the flat-faced type of follower unless very large base circles are used to force \( p \) to be positive everywhere.
8.8 CAM MANUFACTURING CONSIDERATIONS

The preceding sections illustrate that there are a number of factors to consider when designing a cam. A great deal of care in design is necessary to obtain a good compromise of all factors, some of which conflict. Once the cam design is complete a whole new set of considerations must be dealt with that involve manufacturing the cam. After all, if your design cannot be successfully machined in metal in a way that truly represents the theoretical functions chosen, their benefits will not be realized. Unlike linkages, which are very easy to make, cams are a challenge to manufacture properly.

Cams are usually made from strong, hard materials such as medium to high carbon steels (case- or through-hardened) or cast ductile iron or grey cast iron (case-hardened). Cams for low loads and speeds or marine applications are sometimes made of brass or bronze. Even plastic cams are used in such applications as washing machine timers where the cam is merely tripping a switch at the right time. We will concentrate on the higher load-speed situations here for which steel or cast/ductile iron are the only practical choices. These materials range from fairly difficult to very difficult to machine depending on the alloy. At a minimum, a reasonably accurate milling machine is needed to make a cam. A computer controlled machining center is far preferable and is most often the choice for serious cam production.

Cams are typically milled with rotating cutters that in effect "tear" the metal away leaving a less than perfectly smooth surface at a microscopic level. For a better finish and better geometric accuracy, the cam can be ground after milling away most of the unneeded material. Heat treatment is usually necessary to get sufficient hardness to prevent rapid wear. Steel cams are typically hardened to about Rockwell Rc 50-55. Heat treatment introduces some geometric distortion. The grinding is usually done after heat
treatment to correct the contour as well as to improve the finish. The grinding step nearly doubles the cost of an already expensive part, so it is often skipped in order to save money. A hardened but unground cam will have some heat distortion error despite accurate milling before hardening. There are several methods of cam manufacture in common use as shown in Table 8-3.

Geometric Generation

Geometric generation refers to the continuous "sweeping out" of a surface as in turning a cylinder on a lathe. This is perhaps the ideal way to make a cam because it creates a truly continuous surface with an accuracy limited only by the quality of the machine and tools used. Unfortunately there are very few types of cams that can be made by this method. The most obvious one is the eccentric cam (Figure 8-10, p. 359) which can be turned and ground on a lathe. A cycloid can also be geometrically generated. Very few other curves can. The presence of dwells makes it extremely difficult to apply this method. Thus, it is seldom used for cams. However, when it can be, as in the case of the eccentric cam of Figure 8-10, the resulting acceleration, though not perfect, is very close to the theoretical cosine wave as seen in Figure 8-11 (p. 359). This eccentric cam was made by turning and grinding on a high-quality lathe. This is the best that can be obtained in cam manufacture. Note that the displacement function is virtually perfect. The errors are only visible in the more sensitive acceleration function measurement.

Manual or NC Machining to Cam Coordinates (Plunge-Cutting)

Computer-aided manufacturing (CAM) has become the virtual standard for high accuracy machining in the United States. Numerical control (NC) machinery comes in many types. Lathes, milling machines, grinders, etc., are all available with on-board computers which control either the position of the workpiece, the tool, or both. The simplest type of NC machine moves the tool (or workpiece) to a specified $x,y$ location and then drives the tool (say a drill) down through the workpiece to make a hole. This process is repeated as much as necessary to create the part. This simple process is referred to as NC to distinguish it from continuous numerical control (CNC).

This NC process is sometimes used for cam manufacture, and even for master cams as described below. It is, in fact, merely a computerized version of the old manual method of cam milling, which is often called plunge-cutting to refer to plunging the spinning milling cutter down through the workpiece. This is not the best way to machine a cam because it leaves "scallops" on the surface as shown in Figure 8-51, due to the fact that part of the cutting edge is not engaged when it is cut off by a flat surface. The scallops cannot be removed by grinding. If the cam is to be used for automotive camshafts, the last cut should be done with a flat-faced grinding wheel.
that the machinist can only plunge at a discrete number of positions around the cam. In effect, the displacement function which we developed has to be "discretized" or sampled at some finite number of places around the cam. Practicality limits this digitizing process to increments of about 1/2 to 1 degree. With an NC process the increment might be reduced to 1/4 or 1/10 degree. At some point "diminishing returns" will set in as the machine's ability to resolve positions spaced too closely will limit the accuracy. Standard milling machines can be expected to give accuracies in the 0.001 inch tolerance range. Tooling-quality machining centers, jig borers, and grinders can be as much as two to 20 times that accurate [0.0005 in down to 0.000050-in (50 millionths) tolerance].

The scallops that are left on the cam after plunge-cutting have to be removed by hand-dressing with files and grindstones. This obviously introduces more error. Even if the bottoms of the scallops at the sample increments were exactly correct, all points between are subject to the vagaries of hand work. The chance of exactly achieving the designed $s v a j$ functions, especially the higher derivatives, with this manufacturing method is slight.

### Continuous Numerical Control with Linear Interpolation

In a CNC machine, the tool is in constant contact with the workpiece, always cutting, while the computer controls the movement of the workpiece from position to position as stored in its memory. This is a **continuous cutting process** as opposed to the discrete one of NC. However, the cam displacement function must still be discretized or sampled at some angular increment. The common increments are 1/4, 1/2, and 1 degree. Since the machine only has information about the x,y locations of these 360, 720, or 1440 points around the cam, it has to figure out how to get from one point to another while cutting. The most common method used to "fill in" the missing data is **linear interpolation** (LI). The machine's computer calculates the straight line between each pair of data points and then drives the cutter (or workpiece) so as to stay as close to that line as it can. If it could do this perfectly (which it can't), we would get a piecewise continuous first-order approximation to the cam contour. This would introduce slope discontinuities which will create theoretically infinite pulses of acceleration. We would be back to

![Diagram](image)

**Figure 8-51**

Plunge-cutting a cam leaves scallops on its surface
our "bad cam by naive designer" of Example 8-1 having infinite acceleration regardless of the actual function selected.

An improvement can be made in this linear interpolation scheme by fitting a cubic spline curve to the cam coordinate data and then resampling this spline approximation at finer spacing down to the machine's angular resolution. This denser data set is then used to drive the cutter which still must traverse approximate straight-line paths between the close spaced data points. The curve fitting and resampling is typically done at the manufacturing stage.

Fortunately, the dynamics of the cutting process which are a function of speeds, feeds, tool sharpness, tool chatter, deflection of the spindle, etc., all conspire to prevent the formation of a series of distinct "flats" which would give the derivatives shown in Figure 8-8. Rather, the kind of acceleration curve that actually results from a cam which was milled (but not ground) on a very high-quality CNC machining center, using 1 degree linear interpolation, is as shown in Figure 8-52. The program is a simple harmonic eccentric with no dwells. The dynamic curves were measured with instrumentation on the roller follower while the cam was running at 600 rpm on a custom-designed cam dynamic test fixture (CDTF) [8]. The actual displacement is quite true to the theoretical, but the acceleration has a significant amount of vibratory noise present which distorts the function from its theoretical cosine waveshape. The acceleration is shown in g's. Compare the peak-to-peak values in Figure 8-52 (8 g) with the same cam design made with geometric generation in Figure 8-11 (p. 359), (5 g). The error is of the order of 3 g's on a base of 5 g's. These errors in the acceleration are due to a combination of manufacturing factors as described above. It was found that the use of 1/4, 1/2 or 1 degree digitization increments on this size cam made no statistically significant difference in the fidelity of the actual acceleration function to its theoretical waveform.[9]

The physical fidelity of the cam surface of the same 1 degree linear interpolated eccentric cam to a geometrically generated (turned and ground) reference cam can be seen in Figure 8-53 which is an enlarged section of a portion of the cams as measured to

* These cams were about 8 in (200 mm) in diameter. If the cam diameter is larger, then smaller angular increments of digitization will be needed as the distance along the pitch curve between data points for any angular increment increases linearly with prime circle diameter.
O.0005-in accuracy. The axes are calibrated in inches. These figures show that linear interpolated CNC is a reasonably accurate method of cam manufacture.

Continuous Numerical Control with Circular Interpolation

This process is similar to CNC with linear interpolation except that a circular interpolation (CI) algorithm is used between data points. This potentially allows a sparser (thus smaller) database in the machine which can be an advantage. The computer tries to fit a circle arc to as many adjacent data points as possible without exceeding a user-selected error band around the actual displacement function. This reduces the (variable) number of fitted data points for any arc segment to three, a radius and its two center coordinates. Most CNC machines have a built-in algorithm to generate (cut) circle arcs quickly and efficiently, as this is a common requirement in normal machining.

Figure 8-54 shows the same cam design as in Figures 8-11 and 8-52, made on the same machining center from the same bar of steel, but with circular interpolation (CI). The error in acceleration is more than the turned-ground (TG) cam but less than the linear interpolated (LI) one. Figure 8-55 shows the cam contour of the CI cam compared to the TG cam. Based on dynamic performance the circular interpolated cam has lower acceleration error than the linear interpolated cam and the difference is statistically significant. The "blip" in the middle of the period is due to the slight ridge formed at the point where the cutter starts and stops its continuous sweep around the cam contour.

Analog Duplication

The last method listed, analog duplication, involves the creation of a master cam, sometimes at larger than full scale, which is subsequently used in a cam duplicating machine to turn out large quantities of the finished cams. Some automotive camshafts are still
made by this method though CNC methods are replacing the automotive cam duplicating machines. Analog duplication can be the most economical method where production quantities are high.

A cam duplicating machine has two spindles. The master cam is mounted on one, and the workpiece is placed on the other. A dummy cutter mounted on one crank of a pantograph linkage is used as a follower against the master. The actual cutter is mounted on the other crank of the pantograph linkage. A multiplication ratio can be introduced.
between the dummy follower and the cutter to size the finished cam versus the master. This allows master cams of larger size to be used which increases the accuracy. As the master and slave slowly turn together, synchronous and in phase, the dummy cutter follows the master’s contour and the workpiece is cut to match. This process can be done with either milling (cutting) or grinding of the cam surface. Typically the cam is rough cut first, and then heat treated and reground to finished size. Some cams are left as-milled with no post heat-treat grinding.

This analog duplication method can obviously create a cam that is only as good as the master cam at best. Some errors will be introduced in the duplicating process due to deflections of the tool or machine parts, but the quality of the master cam ultimately limits the quality of the finished cams. The master cam is typically made by one of the other methods listed in Table 8-3 (p. 413), each of which has its limitations. The master cam may require some hand-dressing with files or hand grindstones to smooth its surface. A plunge-cut cam requires a lot of hand-dressing, the CNC cams less so. If hand-dressing is done, it will result in a very smooth surface but the chances that the resulting contour is an accurate representation of the designed functions, especially the higher derivatives, is slim. Thus the finished cam may not be an accurate representation of the design.

Figure 8-56 shows the same cam design as in Figures 8-52 (p. 415) and 8-54 (p. 417), made on the same machining center from the same bar of steel, but analog duplicated from a hand-dressed, plunge-cut master. This represents the worst case in terms of manufacturing error. The error in acceleration is more than any of the other cams. Figure 8-57 shows the cam contour of the analog milled cam compared to the reference turned-ground cam. It is much less accurate than either of the CNC versions. Based on dynamic performance, the analog milled cam from a hand-dressed, plunge-cut master has a higher acceleration error than any other cam tested and the difference is statistically significant. If the master cam were made by a more accurate method, the accuracy of the production cam could be better but would still be potentially inferior to one made with direct CNC.

**Actual Cam Performance Compared to Theoretical Performance**

The relative peak accelerations of several common double-dwell cam functions were discussed in Section 8.3 (p. 353) and summarized in Table 8-2 (p. 370). That discussion also emphasized the importance of a smooth jerk function for minimizing vibrations. The theoretical differences between peak accelerations of different cam functions will be altered by the presence of vibratory noise in the actual acceleration waveforms. This noise will be due in part to errors introduced in the manufacturing process, as discussed above, but there will also be inherent differences due to the degree to which the jerk function excites vibrations in the cam-follower train. These vibrations will be heavily influenced by the structural dynamic characteristics of the follower train itself. In general, a very stiff and massive follower train will vibrate less than a light and flexible one, but the presence of frequencies in the cam function that are near the natural frequencies of the follower train will exacerbate the problem.

Figure 8-58 shows the actual acceleration waveforms of four common double-dwell cam programs, modified trapezoid, modified sine, cycloidal, and 4-5-6-7 polynomial, which were ground on the same four-dwell cam. These waveforms were measured with
an accelerometer mounted on the stiff follower train of a cam dynamic test fixture (CDTF) specially designed for low vibration and noise. These test cams were made by a specialty cam manufacturer using linear interpolation CNC with a 1/4 degree digitizing increment on very high quality machining and grinding equipment. After hardening, the cam contour was ground to a surface finish with an average roughness \((R_a)\) of 0.125 microns. Comparing these functions on the same cam, made on the same production machinery, and run against the same follower system gives a means to measure their actual differences in relative accelerations.

\[\text{FIGURE 8-57} \]

Contours of 2 cams: one turned and ground; one milled with analog duplication of a hand-dressed master cam
All four acceleration waveforms have some amount of vibratory noise present which is seen as oscillations or ripples on the curves. Compare these curves to their exact theoretical equivalents in Figure 8-18 (p. 370). Table 8-4 compares the expected theoretical peak-to-peak acceleration $A_{pp\text{theoretical}}$ for each cam (column 2) with its actual measured peak-to-peak acceleration $A_{pp\text{actual}}$ (column 3). The error is expressed in column 4 as an acceleration multiplier factor $A_{nf}$ defined as:

$$A_{nf} = \frac{A_{pp\text{actual}}}{A_{pp\text{theoretical}}}$$

(8.45)

**FIGURE 8-58**

Actual measured follower acceleration curves of four double-dwell programs.
Column five in Table 8-4 shows the maximum acceleration factors for these four functions taken from Table 8-2. The last (sixth) column shows the actual maximum acceleration values based on these test data and is the product of columns four and five.

These values of actual maximum acceleration show smaller differences between the functions than are predicted by their theoretical waveforms. The modified trapezoid, which has the lowest theoretical acceleration, has a 13% noise penalty due to vibration. The modified sine has a 14% noise penalty, while the cycloidal and 4-5-6-7 polynomial functions have only 5 to 6% noise. This is due to the fact that the last two functions have smoother jerk waveforms than the first two. The cycloidal waveform, with its cosine jerk function, is a good choice for high-speed operations. Its acceleration is actually only about 19% greater than the modified trapezoid’s and 5% more than the modified sine, rather than the 28% and 14% differentials predicted by the theoretical peak values.

8.9 PRACTICAL DESIGN CONSIDERATIONS

The cam designer is often faced with many confusing decisions, especially at an early stage of the design process. Many early decisions, often made somewhat arbitrarily and without much thought, can have significant and costly consequences later in the design. The following is a discussion of some of the trade-offs involved with such decisions in the hope that it will provide the cam designer with some guidance in making these decisions.

Translating or Oscillating Follower?

There are many cases, especially early in a design, when either translating or rotating motion could be accommodated as output from the cam. Approximate straight-line motion is often adequate and can be obtained from a large-radius rocker follower. The rocker or oscillating follower has one significant advantage over the translating follower when a roller follower is used. A translating follower is free to rotate about its axis of translation and may need to have some antirotation guiding (such as a keyway) provided to prevent misalignment of a roller follower with the cam.
Conversely, the oscillating follower will keep the roller follower aligned in the same plane as the cam with no guiding beyond its own pivot. Also, the pivot friction in an oscillating follower typically has a small moment arm compared to the moment of the force from the cam on the follower arm. But, the friction force on a translating follower has a one-to-one geometric relationship with the cam force. This can have a larger parasitic effect on the system.

On the other hand, translating flat-faced followers are often deliberately arranged with their axis slightly out of the plane of the cam in order to create a rotation about their own axis due to the frictional moment resulting from the offset. The flat follower will then precess around its own axis and distribute the wear over its entire face surface. This is common practice in automotive valve cams that use flat-faced followers or “tappets.”

**Force or Form-Closed?**

A form-closed (track or groove) cam is more expensive to make than a force-closed (open) cam simply because there are two surfaces to machine and grind. Also, heat treating will often distort the track of a form-closed cam, narrowing or widening it such that the roller follower will not fit properly. This virtually requires post heat-treat grinding for track cams in order to resize the slot. An open (force-closed) cam will also distort on heat-treating but may still be usable without grinding.

**FOLLOWER JUMP** The principal advantage of a form-closed (track) cam is that it does not need a return spring, and thus can be run at higher speeds than a force-closed cam whose spring and follower mass will go into resonance at some speed, causing potentially destructive follower jump. This phenomenon will be investigated in Chapter 15 on cam dynamics. High-speed automobile and motorcycle racing engines often use form-closed (desmodromic) valve cam trains to allow higher engine rpm without incurring valve “float,” or follower jump.

**CROSSOVER SHOCK** Though the lack of a return spring can be an advantage, it comes, as usual, with a trade-off. In a form-closed (track) cam there will be crossover shock each time the acceleration changes sign. Crossover shock describes the impact force that occurs when the follower suddenly jumps from one side of the track to the other as the dynamic force \((ma)\) reverses sign. There is no flexible spring in this system to absorb the force reversal as in the force-closed case. The high impact forces at crossover cause noise, high stresses, and local wear. Also, the roller follower has to reverse direction at each crossover which causes sliding and accelerates follower wear. Studies have shown that roller followers running against a well-lubricated open radial cam have slip rates of less than 1%.

**Radial or Axial Cam?**

This choice is largely dictated by the overall geometry of the machine for which the cam is being designed. If the follower must move parallel to the camshaft axis, then an axial cam is dictated. If there is no such constraint, a radial cam is probably a better choice simply because it is a less complicated, thus cheaper, cam to manufacture.
Roller or Flat-Faced Follower?

The roller follower is a better choice from a earn design standpoint simply because it accepts negative radius of curvature on the earn. This allows more variety in the earn program. Also, for any production quantities, the roller follower has the advantage of being available from several manufacturers in any quantity from one to a million. For low quantities it is not usually economical to design and build your own custom follower. In addition, replacement roller followers can be obtained from suppliers on short notice when repairs are needed. Also, they are not particularly expensive even in small quantities.

Perhaps the largest users of flat-faced followers are automobile engine makers. Their quantities are high enough to allow any custom design they desire. It can be made or purchased economically in large quantity and can be less expensive than a roller follower in that case. Also with engine valve cams, a flat follower can save space over a roller. However, many manufacturers have switched to roller followers in automobile engines to reduce friction and improve fuel mileage. Diesel engines have long used roller followers (tappets) as have racers who "hop-up" engines for high performance.

Cams used in automated production line machinery use stock roller followers almost exclusively. The ability to quickly change a worn follower for a new one taken from the stockroom without losing much production time on the "line" is a strong argument in this environment. Roller followers come in several varieties (see Figure 8-5a, p. 351). They are based on roller or ball bearings. Plain bearing versions are also available for low-noise requirements. The outer surface, which rolls against the earn can be either cylindrical or spherical in shape. The "crown" on the spherical follower is slight, but it guarantees that the follower will ride near the center of a flat earn regardless of the accuracy of alignment of the axes of rotation of earn and follower. If a cylindrical follower is chosen and care is not taken to align the axes of earn and roller follower, the follower will ride on one edge and wear rapidly.

Commercial roller followers are typically made of high carbon alloy steel such as AISI 52100 and hardened to Rockwell Rc 60 - 62. The 52100 alloy is well suited to thin sections that must be heat-treated to a uniform hardness. Because the roller makes many revolutions for each earn rotation, its wear rate may be higher than that of the earn. Chrome plating the follower can markedly improve its life. Chrome is harder than steel at about Rc 70. Steel cams are typically hardened to a range of Rc 50 - 55.

To Dwell or Not to Dwell?

The need for a dwell is usually clear from the problem specifications. If the follower must be held stationary for any time, then a dwell is required. Some earn designers tend to insert dwells in situations where they are not specifically needed for follower stasis, in a mistaken belief that this is preferable to providing a rise-return motion when that is what is really needed. If the designer is attempting to use a double-dwell program in a single-dwell case, then perhaps his or her motivation to "let the vibrations settle out" by providing a "short dwell" at the end of the motion is justified. However, he or she probably should be using another earn program, perhaps a polynomial tailored to the specifications. Taking the acceleration to zero, whether for an instant or for a "short dwell," is generally unnecessary and undesirable. (See Examples 8-5, p. 375, 8-6, p. 377, and 8-8,
p. 383) A dwell should be used only when the follower is required to be stationary for some measurable time. Moreover, if you do not need any dwell at all, consider using a linkage instead. They are a lot easier and cheaper to manufacture.

To Grind or Not to Grind?

Many production machinery cams are used as-milled, and not ground. Automotive valve cams are ground. The reasons are largely due to cost and quantity considerations as well as the high speeds of automotive cams. There is no question that a ground cam is superior to a milled cam. The question in each case is whether the advantage gained is worth the cost. In small quantities, as are typical of production machinery, grinding about doubles the cost of a cam. The advantages in terms of smoothness and quietness of operation, and of wear, are not in the same ratio as the cost difference. A well-machined cam can perform nearly as well as a well-ground cam and better than a poorly ground cam.[8]

Automotive cams are made in large quantity, run at very high speed, and are expected to last for a very long time with minimal maintenance. This is a very challenging specification. It is a great credit to the engineering of these cams that they very seldom fail in 100,000 miles or more of operation. These cams are made on specialized equipment which keeps the cost of their grinding to a minimum.

To Lubricate or Not to Lubricate?

Cams need lots of lubrication. Automotive cams are literally drowned in a flow of engine oil. Many production machine cams run immersed in an oil bath. These are reasonably happy cams. Others are not so fortunate. Cams which operate in close proximity to the product on an assembly machine in which oil would cause contamination of the product (food products, personal products) often are run dry. Camera mechanisms, which are full of linkages and cams, are often run dry. Lubricant would eventually find its way to the film.

Unless there is some good reason to eschew lubrication, a cam-follower should be provided with a generous supply of clean lubricant, preferably a hypoid-type oil containing additives for boundary lubrication conditions. The geometry of a cam-follower joint (half-joins) is among the worst possible from a lubrication standpoint. Unlike a journal bearing, which tends to trap a film of lubricant within the joint, the half joint is continuously trying to squeeze the lubricant out of itself. This can result in a boundary, or mixed boundary/EHD* lubrication state in which some metal-to-metal contact will occur. Lubricant must be continually resupplied to the joint. Another purpose of the liquid lubricant is to remove the heat of friction from the joint. If run dry, significantly higher material temperatures will result, with accelerated wear and possible early failure.

8.10 REFERENCES


* EHD = ElastoHydroDynamic.


8.11 PROBLEMS

Problems DYNACAM and MATRIX may be used to solve these problems or to check your solution where appropriate.

8-1 Figure P8-1 shows the cam and follower from Problem 6-65. Using graphical methods, find and sketch the equivalent fourbar linkage for this position of the cam and follower.

8-2 Figure P8-1 shows the cam and follower from Problem 6-65. Using graphical methods, find the pressure angle at the position shown.

8-3 Figure P8-2 shows a cam and follower. Using graphical methods, find and sketch the equivalent fourbar linkage for this position of the cam and follower.

![Figure P8-1](image_url)
8-4 Figure P8-2 shows a cam and follower. Using graphical methods, find the pressure angle at the position shown.

8-5 Figure P8-3 shows a cam and follower. Using graphical methods, find and sketch the equivalent fourbar linkage for this position of the cam and follower.

8-6 Figure P8-3 shows a cam and follower. Using graphical methods, find the pressure angle at the position shown.

8-7 Design a double-dwell cam to move a follower from 0 to 2.5° in 60°, dwell for 120°, fall 2.5° in 30° and dwell for the remainder. The total cycle must take 4 sec. Choose suitable programs for rise and fall to minimize accelerations. Plot the s v a j diagrams.

8-8 Design a double-dwell cam to move a follower from 0 to 1.5° in 45°, dwell for 150°, fall 1.5° in 90° and dwell for the remainder. The total cycle must take 6 sec. Choose suitable programs for rise and fall to minimize velocities. Plot the s v a j diagrams.

8-9 Design a single-dwell cam to move a follower from 0 to 2° in 60°, fall 2° in 90° and dwell for the remainder. The total cycle must take 2 sec. Choose suitable programs for rise and fall to minimize accelerations. Plot the s v a j diagrams.

8-10 Design a three-dwell cam to move a follower from 0 to 2.5° in 40°, dwell for 100°, fall 1.5° in 90°, dwell for 20°, fall 1° in 30° and dwell for the remainder. The total cycle must take 10 sec. Choose suitable programs for rise and fall to minimize velocities. Plot the s v a j diagrams.

8-11 Design a four-dwell cam to move a follower from 0 to 2.5° in 40°, dwell for 100°, fall 1.5° in 90°, dwell for 20°, fall 0.5° in 30°, dwell for 40°, fall 0.5° in 30° and dwell for the remainder. The total cycle must take 15 sec. Choose suitable programs for rise and fall to minimize accelerations. Plot the s v a j diagrams.

8-12 Size the cam from Problem 8-7 for a 1° radius roller follower considering pressure angle and radius of curvature. Use eccentricity only if necessary to balance those functions. Plot both those functions. Draw the cam profile. Repeat for a flat-faced follower. Which would you use?

8-13 Size the cam from Problem 8-8 for a 1.5° radius roller follower considering pressure angle and radius of curvature. Use eccentricity only if necessary to balance those
functions. Plot both those functions. Draw the cam profile. Repeat for a flat-faced follower. Which would you use?

8-15 Size the cam from Problem 8-10 for a 2" radius roller follower considering pressure angle and radius of curvature. Use eccentricity only if necessary to balance those functions. Plot both those functions. Draw the cam profile. Repeat for a flat-faced follower. Which would you use?

8-16 Size the cam from Problem 8-11 for a 0.5" radius roller follower considering pressure angle and radius of curvature. Use eccentricity only if necessary to balance those functions. Plot both those functions. Draw the cam profile. Repeat for a flat-faced follower. Which would you use?

8-17 A high friction, high inertia load is to be driven. We wish to keep peak velocity low. Combine segments of half period cycloidal displacements with a constant velocity segment on both rise and fall to reduce the maximum velocity below that obtainable with a full period modified sine acceleration alone (i.e., with no constant velocity portion). Rise is 1" in 90°, dwell for 60°, fall in 30°, dwell for remainder. Compare the two designs and comment. Use an α of one for comparison.

8-18 A constant velocity of 0.4 in/sec must be matched for 1.5 sec. Then the follower must return to your choice of start point and dwell for 2 sec. The total cycle time is 6 sec. Design a cam for a follower radius of 0.75" and a maximum pressure angle of 30° absolute value.

8-19 A constant velocity of 0.25 in/sec must be matched for 3 sec. Then the follower must return to your choice of start point and dwell for 3 sec. The total cycle time is 12 sec. Design a cam for a follower radius of 1.25" and a maximum pressure angle of 35° absolute value.

8-20 A constant velocity of 2 in/sec must be matched for 1 second. Then the follower must return to your choice of start point. The total cycle time is 2.75 sec. Design a cam for a follower radius of 0.5" and a maximum pressure angle of 25° absolute value.

8-21 Write a computer program or use an equation solver such as Mathcad or TKSolver to calculate and plot the s v a j diagrams for a modified trapezoidal acceleration cam function for any specified values of lift and duration. Test it using a lift of 20 mm over 60° at 1 rad/sec.

8-22 Write a computer program or use an equation solver such as Mathcad or TKSolver to calculate and plot the s v a j diagrams for a modified sine acceleration cam function for any specified values of lift and duration. Test it using a lift of 20 mm over 60° at 1 rad/sec.

8-23 Write a computer program or use an equation solver such as Mathcad or TKSolver to calculate and plot the s v a j diagrams for a cycloidal displacement cam function for any specified values of lift and duration. Test it using a lift of 20 mm over 60° at 1 rad/sec.

8-24 Write a computer program or use an equation solver such as Mathcad or TKSolver to calculate and plot the s v a j diagrams for a 3-4-5 polynomial displacement cam

† These problems are suited to solution using Mathcad, Matlab, or TKSolver equation solver programs.

‡ These problems are suited to solution using program DYNACAM, which is on the attached CD-ROM.
function for any specified values of lift and duration. Test it using a lift of 20 mm over 60° at 1 rad/sec.

8-25 Write a computer program or use an equation solver such as Mathcad or TK Solver to calculate and plot the s-v-a-j diagrams for a 4-5-6-7 polynomial displacement cam function for any specified values of lift and duration. Test it using a lift of 20 mm over 60° at 1 rad/sec.

8-26 Write a computer program or use an equation solver such as Mathcad or TK Solver to calculate and plot the s-v-a-j diagrams for a simple harmonic displacement cam function for any specified values of lift and duration. Test it using a lift of 20 mm over 60° at 1 rad/sec.

8-27 Write a computer program or use an equation solver such as Mathcad or TK Solver to calculate and plot the pressure angle and radius of curvature for a modified trapezoidal acceleration cam function for any specified values of lift, duration, eccentricity, and prime circle radius. Test it using a lift of 20 mm over 60° at 1 rad/sec, and determine the prime circle radius needed to obtain a maximum pressure angle of 20°. What is the minimum diameter of roller follower needed to avoid undercutting with these data?

8-28 Write a computer program or use an equation solver such as Mathcad or TK Solver to calculate and plot the pressure angle and radius of curvature for a modified sine acceleration cam function for any specified values of lift, duration, eccentricity, and prime circle radius. Test it using a lift of 20 mm over 60° at 1 rad/sec, and determine the prime circle radius needed to obtain a maximum pressure angle of 20°. What is the minimum diameter of roller follower needed to avoid undercutting with these data?

8-29 Write a computer program or use an equation solver such as Mathcad or TK Solver to calculate and plot the pressure angle and radius of curvature for a cycloidal displacement cam function for any specified values of lift, duration, eccentricity, and prime circle radius. Test it using a lift of 20 mm over 60° at 1 rad/sec, and determine the prime circle radius needed to obtain a maximum pressure angle of 20°. What is the minimum diameter of roller follower needed to avoid undercutting with these data?

8-30 Write a computer program or use an equation solver such as Mathcad or TK Solver to calculate and plot the pressure angle and radius of curvature for a 3-4-5 polynomial displacement cam function for any specified values of lift, duration, eccentricity, and prime circle radius. Test it using a lift of 20 mm over 60° at 1 rad/sec, and determine the prime circle radius needed to obtain a maximum pressure angle of 20°. What is the minimum diameter of roller follower needed to avoid undercutting with these data?

8-31 Write a computer program or use an equation solver such as Mathcad or TK Solver to calculate and plot the pressure angle and radius of curvature for a 4-5-6-7 polynomial displacement cam function for any specified values of lift, duration, eccentricity, and prime circle radius. Test it using a lift of 20 mm over 60° at 1 rad/sec, and determine the prime circle radius needed to obtain a maximum pressure angle of 20°. What is the minimum diameter of roller follower needed to avoid undercutting with these data?

8-32 Write a computer program or use an equation solver such as Mathcad or TK Solver to calculate and plot the pressure angle and radius of curvature for a simple harmonic displacement cam function for any specified values of lift, duration, eccentricity, and