4.1 INTRODUCTION

It is often important to determine the force produced on a solid body by fluid flowing steadily over or through it. For example, there is the force exerted on a solid surface by a jet of fluid impinging on it; there are also the aerodynamic forces (lift and drag) on an aircraft wing, the force on a pipe-bend caused by the fluid flowing within it, the thrust on a propeller and so on. All these forces are associated with a change in the momentum of the fluid.

The magnitude of such a force is determined essentially by Newton’s Second Law. However, the law usually needs to be expressed in a form particularly suited to the steady flow of a fluid: this form is commonly known as the steady-flow momentum equation and may be applied to the whole bulk of fluid within a prescribed space. Only forces acting at the boundaries of this fluid concern us; any force within this fluid is involved only as one half of an action-and-reaction pair and so does not affect the overall behaviour. Moreover, the fluid may be compressible or incompressible, and the flow with or without friction.

4.2 THE MOMENTUM EQUATION FOR STEADY FLOW

In its most general form, Newton’s Second Law states that the net force acting on a body in any fixed direction is equal to the rate of increase of momentum of the body in that direction. Since force and momentum are both vector quantities it is essential to specify the direction. Where we are concerned with a collection of bodies (which we shall here term a system) the law may be applied (for a given direction) to each body individually. If the resulting equations are added, the total force in the given fixed direction corresponds to the net force acting in that direction at the boundaries of the system. Only these external, boundary forces are involved because any internal forces between the separate bodies occur in pairs of action and reaction and therefore cancel in the total. For a fluid, which is continuum of particles, the same result applies: the net force in any fixed direction on a certain defined amount of fluid equals the total rate of increase of momentum of that fluid in that direction.
Our aim now is to derive a relation by which force may be related to the fluid within a given space. We begin by applying Newton’s Second Law to a small element in a stream-tube (shown in Fig. 4.1). The flow is steady and so the stream-tube remains stationary with respect to the fixed coordinate axes. The cross-section of this stream-tube is sufficiently small for the velocity to be considered uniform over the plane \(AB\) and over the plane \(CD\). After a short interval of time \(\delta t\) the fluid that formerly occupied the space \(ABCD\) will have moved forward to occupy the space \(A'B'C'D'\). In general, its momentum changes during this short time interval.

If \(u_x\) represents the component of velocity in the \(x\) direction then the element (of mass \(\delta m\)) has a component of momentum in the \(x\) direction equal to \(u_x\delta m\). The total \(x\)-momentum of the fluid in the space \(ABCD\) at the beginning of the time interval \(\delta t\) is therefore

\[
\sum_{ABCD} u_x \delta m
\]

The same fluid at a time \(\delta t\) later will have a total \(x\)-momentum

\[
\sum_{A'B'C'D'} u_x \delta m
\]

The last expression may be expanded as

\[
\sum_{ABCD} u_x \delta m - \sum_{ABB'A'} u_x \delta m + \sum_{DCC'D'} u_x \delta m
\]

The net increase of \(x\)-momentum during the time interval \(\delta t\) is therefore

\[
\left( \sum_{A'B'C'D'} u_x \delta m \right) \text{ after } \delta t - \left( \sum_{ABCD} u_x \delta m \right) \text{ before } \delta t
\]
The momentum equation

\[
\begin{align*}
= \left( \sum_{ABCD} u_x \delta m - \sum_{ABB'A'} u_x \delta m + \sum_{DCC'D'} u_x \delta m \right)_{\text{after } \delta t} \\
- \left( \sum_{ABCD} u_x \delta m \right)_{\text{before } \delta t} \\
= \left( \sum_{DCC'D'} u_x \delta m - \sum_{ABB'A'} u_x \delta m \right)_{\text{after } \delta t}
\end{align*}
\]

since, as the flow is assumed steady, \((\sum u_x \delta m)_{ABCD}\) is the same after \(\delta t\) as before \(\delta t\). Thus, during the time interval \(\delta t\), the increase of \(x\)-momentum of the batch of fluid considered is equal to the \(x\)-momentum leaving the stream-tube in that time minus the \(x\)-momentum entering in that time:

\[
\left( \sum_{DCC'D'} u_x \delta m \right) - \left( \sum_{ABB'A'} u_x \delta m \right)
\]

For a very small value of \(\delta t\) the distances \(AA', BB'\) are very small, so the values of \(u_x\), for all the particles in the space \(ABB'A'\) are substantially the same. Similarly, all particles in the space \(DCC'D'\) have substantially the same value of \(u_x\), although this may differ considerably from the value for particles in \(ABB'A'\). The \(u_x\) terms may consequently be taken outside the summations.

Therefore the increase of \(x\)-momentum during the interval \(\delta t\) is

\[
\left( u_x \sum \delta m \right)_{DCC'D'} - \left( u_x \sum \delta m \right)_{ABB'A'} \tag{4.1}
\]

Now \((\sum \delta m)_{DCC'D'}\) is the mass of fluid which has crossed the plane \(CD\) during the interval \(\delta t\) and so is expressed by \(\dot{m}\delta t\), where \(\dot{m}\) denotes the rate of mass flow. Since the flow is steady, \((\sum \delta m)_{ABB'A'}\) also equals \(\dot{m}\delta t\). Thus expression 4.1 may be written \(\dot{m}(u_{x2} - u_{x1})\delta t\), where suffix 1 refers to the inlet section of the stream-tube, suffix 2 to the outlet section. The rate of increase of \(x\)-momentum is obtained by dividing by \(\delta t\), and the result, by Newton’s Second Law, equals the net force \(F_x\) on the fluid in the \(x\) direction

\[
F_x = \dot{m}(u_{x2} - u_{x1}) \tag{4.2}
\]

The corresponding force in the \(x\) direction exerted by the fluid on its surroundings is, by Newton’s Third Law, \(-F_x\).

A similar analysis for the relation between force and rate of increase of momentum in the \(y\) direction gives

\[
F_y = \dot{m}(u_{y2} - u_{y1}) \tag{4.3}
\]

In steady flow \(\dot{m}\) is constant and so \(\dot{m} = \varrho A_1 u_1 = \varrho A_2 u_2\) where \(\varrho\) represents the density of the fluid and \(A\) the cross-sectional area of the stream-tube (\(A\) being perpendicular to \(u\)).

We have so far considered only a single stream-tube with a cross-sectional area so small that the velocity over each end face \((AB, CD)\) may be considered uniform. Let us now consider a bundle of adjacent stream-tubes, each of cross-sectional area \(\delta A\), which together carry all the flow being examined.
The momentum equation for steady flow

The velocity, in general, varies from one stream-tube to another. The space enclosing all these stream-tubes is often known as the control volume and it is to the boundaries of this volume that the external forces are applied. For one stream-tube the ‘x-force’ is given by

$$\delta F_x = \dot{m}(u_{x_2} - u_{x_1}) = \varrho_2 \delta A_2 u_{x_2} - \varrho_1 \delta A_1 u_{x_1}$$

The total force in the x direction is therefore

$$F_x = \int dF_x = \int \varrho_2 u_{x_2} dA_2 - \int \varrho_1 u_{x_1} dA_1$$  \hspace{1cm} (4.4a)

(The elements of area $\delta A$ must everywhere be perpendicular to the velocities $u$.) Similarly

$$F_y = \int \varrho_2 u_{y_2} dA_2 - \int \varrho_1 u_{y_1} dA_1$$  \hspace{1cm} (4.4b)

and

$$F_z = \int \varrho_2 u_{z_2} dA_2 - \int \varrho_1 u_{z_1} dA_1$$  \hspace{1cm} (4.4c)

These equations are required whenever the force exerted on a flowing fluid has to be calculated. They express the fact that for steady flow the net force on the fluid in the control volume equals the net rate at which momentum flows out of the control volume, the force and the momentum having the same direction. It will be noticed that conditions only at inlet 1 and outlet 2 are involved. The details of the flow between positions 1 and 2 therefore do not concern us for this purpose. Such matters as friction between inlet and outlet, however, may alter the magnitudes of quantities at outlet.

It will also be noticed that eqns 4.4 take account of variation of $\varrho$ and so are just as applicable to the flow of compressible fluids as to the flow of incompressible ones.

The integration of the terms on the right-hand side of the eqns 4.4 requires information about the velocity profile at sections 1 and 2. By judicious choice of the control volume, however, it is often possible to use sections 1 and 2 over which $\varrho, u, u_x$ and so on do not vary significantly, and then the equations reduce to one-dimensional forms such as:

$$F_x = \varrho_2 u_2 A_2 - \varrho_1 u_1 A_1 = \dot{m}(u_2 - u_1)$$

It should never be forgotten, however, that this simplified form involves the assumption of uniform values of the quantities over the inlet and outlet cross-sections of the control volume: the validity of these assumptions should therefore always be checked. (See Section 4.2.1.)

A further assumption is frequently involved in the calculation of $F$. A contribution to the total force acting at the boundaries of the control volume comes from the force due to the pressure of the fluid at a cross-section of the flow. If the streamlines at this cross-section are sensibly straight and parallel, the pressure over the section varies uniformly with depth as for a static fluid; in other words, $p^* = p + \varrho gz$ is constant. If, however, the streamlines are not straight and parallel, there are accelerations perpendicular to them and consequent variations of $p^*$. Ideally, then, the control volume should be so selected that at the sections where fluid enters or leaves it the
streamlines are sensibly straight and parallel and, for simplicity, the density and the velocity (in both magnitude and direction) should be uniform over the cross-section.

Newton’s Laws of Motion, we remember, are limited to describing motions with respect to coordinate axes that are not themselves accelerating. Consequently the momentum relations for fluids, being derived from these Laws, are subject to the same limitation. That is to say, the coordinate axes used must either be at rest or moving with uniform velocity in a straight line.

Here we have developed relations only for steady flow in a stream-tube. More general expressions are beyond the scope of this book.

4.2.1 Momentum correction factor

By methods analogous to those of Section 3.5.3 it may be shown that where the velocity of a constant-density fluid is not uniform (although essentially parallel) over a cross-section, the true rate of momentum flow perpendicular to the cross-section is not $\rho u^2 A$ but $\rho \int u^2 dA = \beta \rho u^2 A$. Here $\bar{u} = (1/A) \int u dA$, the mean velocity over the cross-section, and $\beta$ is the momentum correction factor. Hence

$$\beta = \frac{1}{A} \int_A \left( \frac{u}{\bar{u}} \right)^2 dA$$

It should be noted that the velocity $u$ must always be perpendicular to the element of area $dA$. With constant $\rho$, the value of $\beta$ for the velocity distribution postulated in Section 3.5.3 is $100/98 = 1.02$ which for most purposes differs negligibly from unity. Disturbances upstream, however, may give a markedly higher value. For fully developed laminar flow in a circular pipe (see Section 6.2) $\beta = 4/3$. For a given velocity profile $\beta$ is always less than $\alpha$, the kinetic energy correction factor.

4.3 APPLICATIONS OF THE MOMENTUM EQUATION

4.3.1 The force caused by a jet striking a surface

When a steady jet strikes a solid surface it does not rebound from the surface as a rubber ball would rebound. Instead, a stream of fluid is formed which moves over the surface until the boundaries are reached, and the fluid then leaves the surface tangentially. (It is assumed that the surface is large compared with the cross-sectional area of the jet.)

Consider a jet striking a large plane surface as shown in Fig. 4.2. A suitable control volume is that indicated by dotted lines on the diagram. If the $x$ direction is taken perpendicular to the plane, the fluid, after passing over the surface, will have no component of velocity and therefore no momentum in the $x$ direction. (It is true that the thickness of the stream changes as the fluid moves over the surface, but this change of thickness corresponds to a negligible movement in the $x$ direction.) The rate at which $x$-momentum enters
the control volume is \( \int \rho_1 u_1 u_{s1} \, dA_1 = \cos \theta \int \rho_1 u_1^2 \, dA_1 \) and so the rate of increase of \( x \)-momentum is \(- \cos \theta \int \rho_1 u_1^2 \, dA_1 \) and this equals the net force on the fluid in the \( x \) direction. If the fluid on the solid surface were stationary and at atmospheric pressure there would of course be a force between the fluid and the surface due simply to the static (atmospheric) pressure of the fluid. However, the change of fluid momentum is produced by a fluid-dynamic force additional to this static force. By regarding atmospheric pressure as zero we can determine the fluid-dynamic force directly.

Since the pressure is atmospheric both where the fluid enters the control volume and where it leaves, the fluid-dynamic force on the fluid can be provided only by the solid surface (effects of gravity being neglected). The fluid-dynamic force exerted by the fluid on the surface is equal and opposite to this and is thus \( \cos \theta \int \rho_1 u_1^2 \, dA_1 \) in the \( x \) direction. If the jet has uniform density and velocity over its cross-section the integral becomes

\[
\rho_1 u_1^2 \cos \theta \int \, dA_1 = \rho_1 Q_1 u_1 \cos \theta
\]

(where \( Q_1 \) is the volume flow rate at inlet).

The rate at which \( y \)-momentum enters the control volume is equal to \( \sin \theta \int \rho_1 u_1^2 \, dA_1 \). For this component to undergo a change, a net force in the \( y \) direction would have to be applied to the fluid. Such a force, being parallel to the surface, would be a shear force exerted by the surface on the fluid. For an inviscid fluid moving over a smooth surface no shear force is possible, so the component \( \sin \theta \int \rho_1 u_1^2 \, dA_1 \) would be unchanged and equal to the rate at which \( y \)-momentum leaves the control volume. Except when \( \theta = 0 \), the spreading of the jet over the surface is not symmetrical, and for a real fluid the rate at which \( y \)-momentum leaves the control volume differs from the rate at which it enters. In general, the force in the \( y \) direction may be calculated if the final velocity of the fluid is known. This, however, requires further experimental data.

When the fluid flows over a curved surface, similar techniques of calculation may be used as the following example will show.
Example 4.1  A jet of water flows smoothly on to a stationary curved vane which turns it through 60°. The initial jet is 50 mm in diameter, and the velocity, which is uniform, is 36 m·s⁻¹. As a result of friction, the velocity of the water leaving the surface is 30 m·s⁻¹. Neglecting gravity effects, calculate the hydrodynamic force on the vane.

Solution

Taking the x direction as parallel to the initial velocity (Fig. 4.3) and assuming that the final velocity is uniform, we have

Force on fluid in x direction

\[ \text{Rate of increase of } x\text{-momentum} = \varrho Qu_2 \cos 60° - \varrho Qu_1 \]
\[ = (1000 \text{ kg} \cdot \text{m}^{-3}) \left\{ \frac{\pi}{4} (0.05)^2 \text{ m}^2 \times 36 \text{ m} \cdot \text{s}^{-1} \right\} \times (30 \cos 60° \text{ m} \cdot \text{s}^{-1} - 36 \text{ m} \cdot \text{s}^{-1}) \]
\[ = -1484 \text{ N} \]

Similarly, force on fluid in y direction

\[ = \varrho Qu_2 \sin 60° - 0 \]
\[ = \left\{ 1000 \frac{\pi}{4} (0.05)^2 \text{ kg} \cdot \text{s}^{-1} \right\} (30 \sin 60° \text{ m} \cdot \text{s}^{-1}) \]
\[ = 1836 \text{ N} \]

Fig. 4.3

The resultant force on the fluid is therefore \( \sqrt{(-1484^2 + 1836^2)} \) N
\[ = 2361 \text{ N} \]
acting in a direction arctan \( \left\{ \frac{1836}{(-1484)} \right\} = 180° - 51.05° \) to the x direction. Since the pressure is atmospheric both where the fluid enters the control volume and where it leaves, the force on the fluid can be provided only by the vane. The force exerted by the
fluid on the vane is opposite to the force exerted by the vane on the fluid.

Therefore the fluid-dynamic force $F$ on the vane acts in the direction shown on the diagram.

If the vane is moving with a uniform velocity in a straight line the problem is not essentially different. To meet the condition of steady flow (and only to this does the equation apply) coordinate axes moving with the vane must be selected. Therefore the velocities concerned in the calculation are velocities relative to these axes, that is, relative to the vane. The volume flow rate $Q$ must also be measured relative to the vane. As a simple example we may suppose the vane to be moving at velocity $c$ in the same direction as the jet. If $c$ is greater than $u_1$, that is, if the vane is receding from the orifice faster than the fluid is, no fluid can act on the vane at all. If, however, $c$ is less than $u_1$, the mass of fluid reaching the vane in unit time is given by $\rho A(u_1 - c)$ where $A$ represents the cross-sectional area of the jet, and uniform jet velocity and density are assumed. (Use of the relative incoming velocity $u_1 - c$ may also be justified thus. In a time interval $\delta t$ the vane moves a distance $c\delta t$, so the jet lengthens by the same amount; as the mass of fluid in the jet increases by $\rho Ac\delta t$ the mass actually reaching the vane is only $\rho A(u_1 \delta t - Ac\delta t)$ that is, the rate at which the fluid reaches the vane is $\rho A(u_1 - c)$. The direction of the exit edge of the vane corresponds to the direction of the velocity of the fluid there relative to the vane.

The action of a stream of fluid on a single body moving in a straight line has little practical application. To make effective use of the principle a number of similar vanes may be mounted round the circumference of a wheel so that they are successively acted on by the fluid. In this case, the system of vanes as a whole is considered. No longer does the question arise of the jet lengthening so that not all the fluid from the orifice meets a vane; the entire mass flow rate $\rho A u_1$, from the orifice is intercepted by the system of vanes. Such a device is known as a turbine, and we shall consider it further in Chapter 13.

4.3.2 Force caused by flow round a pipe-bend

When the flow is confined within a pipe the static pressure may vary from point to point and forces due to differences of static pressure must be taken into account. Consider the pipe-bend illustrated in Fig. 4.4 in which not only the direction of flow but also the cross-sectional area is changed. The control volume selected is that bounded by the inner surface of the pipe and sections 1 and 2. For simplicity we here assume that the axis of the bend is in the horizontal plane: changes of elevation are thus negligible; moreover, the weights of the pipe and fluid act in a direction perpendicular to this plane and so do not affect the changes of momentum. We assume too that conditions at sections 1 and 2 are uniform and that the streamlines there are straight and parallel.
The momentum equation

If the mean pressure and cross-sectional area at section 1 are \( p_1 \) and \( A_1 \) respectively, the fluid adjacent to this cross-section exerts a force \( p_1 A_1 \) on the fluid in the control volume. Similarly, there is a force \( p_2 A_2 \) acting at section 2 on the fluid in the control volume. Let the pipe-bend exert a force \( F \) on the fluid, with components \( F_x \) and \( F_y \), in the \( x \) and \( y \) directions indicated. The force \( F \) is the resultant of all forces acting over the inner surface of the bend. Then the total force in the \( x \) direction on the fluid in the control volume is

\[
p_1 A_1 - p_2 A_2 \cos \theta + F_x
\]

This total ‘\( x \)-force’ must equal the rate of increase of \( x \)-momentum

\[
\rho Q (u_2 \cos \theta - u_1)
\]

Equating these two expressions enables \( F_x \) to be calculated.

Similarly, the total \( y \)-force acting on the fluid in the control volume is

\[
-p_2 A_2 \sin \theta + F_y = \rho Q (u_2 \sin \theta - 0)
\]

and \( F_y \) may thus be determined. From the components \( F_x \) and \( F_y \) the magnitude and direction of the total force exerted by the bend on the fluid can readily be calculated. The force exerted by the fluid on the bend is equal and opposite to this.

If the bend were empty (except for atmospheric air at rest) there would be a force exerted by the atmosphere on the inside surfaces of the bend. In practice we are concerned with the amount by which the force exerted by the moving fluid exceeds the force that would be exerted by a stationary atmosphere. Thus we use gauge values for the pressures \( p_1 \) and \( p_2 \) in the above equations. The force due to the atmospheric part of the pressure is counterbalanced by the atmosphere surrounding the bend: if absolute values were used for \( p_1 \) and \( p_2 \) separate account would have to be taken of the force, due to atmospheric pressure, on the outer surface.

Where only one of the pressure \( p_1 \) and \( p_2 \) is included in the data of the problem, the other may be deduced from the energy equation.

Particular care is needed in determining the signs of the various terms in the momentum equation. It is again emphasized that the principle used is
that the resultant force on the fluid in a particular direction is equal to the rate of increase of momentum in that direction.

The force on a bend tends to move it and a restraint must be applied if movement is to be prevented. In many cases the joints are sufficiently strong for that purpose, but for large pipes (e.g. those used in hydroelectric installations) large concrete anchorages are usually employed to keep the pipe-bends in place.

The force $F$ includes any contribution made by friction forces on the boundaries. Although it is not necessary to consider friction forces separately they do influence the final result, because they affect the relation between $p_1$ and $p_2$.

**Example 4.2** A 45° reducing pipe-bend (in a horizontal plane) tapers from 600 mm diameter at inlet to 300 mm diameter at outlet (see Fig. 4.5). The gauge pressure at inlet is 140 kPa and the rate of flow of water through the bend is 0.425 m$^3$·s$^{-1}$. Neglecting friction, calculate the net resultant horizontal force exerted by the water on the bend.

**Solution**
Assuming uniform conditions with straight and parallel streamlines at inlet and outlet, we have:

$$u_1 = \frac{0.45 \text{ m}^3\cdot\text{s}^{-1}}{\frac{\pi}{4}(0.6 \text{ m})^2} = 1.503 \text{ m} \cdot \text{s}^{-1}$$

$$u_2 = \frac{0.425 \text{ m}^3\cdot\text{s}^{-1}}{\frac{\pi}{4}(0.3 \text{ m})^2} = 6.01 \text{ m} \cdot \text{s}^{-1}$$

By the energy equation

$$p_2 = p_1 + \frac{1}{2} \rho \left( u_1^2 - u_2^2 \right)$$

$$= 1.4 \times 10^5 \text{ Pa} + 500 \text{ kg} \cdot \text{m}^{-3}(1.503^2 - 6.01^2) \text{ m}^2 \cdot \text{s}^{-2}$$

$$= 1.231 \times 10^5 \text{ Pa}$$

- Fig. 4.5
In the \( x \) direction, force on water in control volume
\[
= p_1 A_1 - p_2 A_2 \cos 45^\circ + F_x = \varrho Q(u_2 \cos 45^\circ - u_1)
\]
= Rate of increase of \( x \)-momentum

where \( F_x \) represents \( x \)-component of force exerted by bend on water. Therefore
\[
1.4 \times 10^5 \text{ Pa} \frac{\pi}{4} 0.62^2 \text{ m}^2 - 1.231 \times 10^5 \text{ Pa} \frac{\pi}{4} 0.32^2 \text{ m}^2 \cos 45^\circ + F_x
= 1000 \text{ kg} \cdot \text{m}^{-3} 0.425 \text{ m}^3 \cdot \text{s}^{-1} (6.01 \cos 45^\circ - 1.503) \text{ m} \cdot \text{s}^{-1}
\]
that is \((39\,580 - 61\,50)\) N + \( F_x \) = 1168 N whence \( F_x = -32\,260 \) N.

In the \( y \) direction, force on water in control volume
\[
= -p_2 A_2 \sin 45^\circ + F_y = \varrho Q(u_2 \sin 45^\circ - 0)
\]
= Rate of increase of \( y \)-momentum, whence
\[
F_y = 1000 \times 0.425(60.1 \sin 45^\circ) \text{ N} + 1.231 \times 10^5 \frac{\pi}{4} 0.32 \sin 45^\circ \text{ N}
= 7960 \text{ N}
\]

Therefore total net force exerted on water = \( \sqrt{(32\,260^2 + 7960^2)} \) N = 33\,230 N acting in direction \( \arctan \{7960/(-32\,260)\} = 180^\circ - 13.86^\circ \) to the \( x \) direction.

Force \( F \) exerted on bend is equal and opposite to this, that is, in the direction shown on Fig. 4.5.

For a pipe-bend with a centre-line not entirely in the horizontal plane the weight of the fluid in the control volume contributes to the force causing the momentum change. It will be noted, however, that detailed information is not required about the shape of the bend or the conditions between the inlet and outlet sections.

4.3.3 Force at a nozzle and reaction of a jet

As a special case of the foregoing we may consider the horizontal nozzle illustrated in Fig. 4.6. Assuming uniform conditions with streamlines straight and parallel at the sections 1 and 2 we have:

Force exerted in the \( x \) direction on the fluid between planes 1 and 2
\[
= p_1 A_1 - p_2 A_2 + F_x = \varrho Q(u_2 - u_1)
\]

If a small jet issues from a reservoir large enough for the velocity within it to be negligible (except close to the orifice) then the velocity of the fluid is increased from zero in the reservoir to \( u \) at the vena contracta (see Fig. 4.7). Consequently the force exerted \( on \) the fluid to cause this change is \( \varrho Q(u - 0) = \varrho QC_v \sqrt{(2gb)} \). An equal and opposite reaction force is therefore exerted by the jet on the reservoir.
The existence of the reaction may be explained in this way. At the vena contracta the pressure of the fluid is reduced to that of the surrounding atmosphere and there is also a smaller reduction of pressure in the neighbourhood of the orifice, where the velocity of the fluid becomes appreciable. On the opposite side of the reservoir, however, and at the same depth, the pressure is expressed by $\rho gh$ and the difference of pressure between the two sides of the reservoir gives rise to the reaction force.

Such a reaction force may be used to propel a craft – aircraft, rocket, ship or submarine – to which the nozzle is attached. The jet may be formed by the combustion of gases within the craft or by the pumping of fluid through it. For the steady motion of such a craft in a straight line the propelling force may be calculated from the momentum equation. For steady flow the reference axes must move with the craft, so all velocities are measured relative to the craft. If fluid (e.g. air) is taken in at the front of the craft with a uniform velocity $c$ and spent fluid (e.g. air plus fuel) is ejected at the rear with a velocity $u_r$ then, for a control volume closely surrounding the craft,

The net rate of increase of fluid momentum backwards (relative to the craft) is

$$\int \rho u_r^2 dA_2 - \int \rho c^2 dA_1$$

(4.5)

where $A_1, A_2$ represent the cross-sectional areas of the entry and exit orifices respectively. (In some jet-propelled boats the intake faces downwards in the bottom of the craft, rather than being at the front. This, however, does not affect the application of the momentum equation since, wherever the water is taken in, the rate of increase of momentum relative to the boat is $\rho Qc$. Nevertheless, a slightly better efficiency can be expected with
a forward-facing inlet because the pressure there is increased – as in a Pitot tube – so the pump has to do less work to produce a given outlet jet velocity.)

Equation 4.5 is restricted to a craft moving steadily in a straight line because Newton’s Second Law is valid only for a non-accelerating set of reference axes.

In practice the evaluation of the integrals in eqn 4.5 is not readily accomplished because the assumption of a uniform velocity – particularly over the area $A_2$ – is seldom justified. Moreover, the tail pipe is not infrequently of diverging form and thus the velocity of the fluid is not everywhere perpendicular to the cross-section.

In a jet-propelled aircraft the spent gases are ejected to the surroundings at high velocity – usually greater than the velocity of sound in the fluid. Consequently (as we shall see in Chapter 11) the pressure of the gases at discharge does not necessarily fall immediately to the ambient pressure. If the mean pressure $p_2$ at discharge is greater than the ambient pressure $p_a$ then a force $(p_2 - p_a)A_2$ contributes to the propulsion of the aircraft.

The relation 4.5 represents the propulsive force exerted by the engine on the fluid in the backward direction. There is a corresponding forward force exerted by the fluid on the engine, and the total thrust available for propelling the aircraft at uniform velocity is therefore

$$(p_2 - p_a)A_2 + \int \rho u_r^2 dA_2 - \int \rho c^2 dA_1$$

(4.6)

It might appear from this expression that, to obtain a high value of the total thrust, a high value of $p_2$ is desirable. When the gases are not fully expanded (see Chapter 11), however, that is, when $p_2 > p_a$, the exit velocity $u_r$ relative to the aircraft is reduced and the total thrust is in fact decreased. This is a matter about which the momentum equation itself gives no information and further principles must be drawn upon to decide the optimum design of a jet-propulsion unit.

A rocket is driven forward by the reaction of its jet. The gases constituting the jet are produced by the combustion of a fuel and appropriate oxidant; no air is required, so a rocket can operate satisfactorily in a vacuum. The penalty of this independence of the atmosphere, however, is that a large quantity of oxidant has to be carried along with the rocket. At the start of a journey the fuel and oxidant together form a large proportion of the total load carried by the rocket. Work done in raising the fuel and oxidant to a great height before they are burnt is wasted. Therefore the most efficient use of the materials is achieved by accelerating the rocket to a high velocity in a short distance. It is this period during which the rocket is accelerating that is of principal interest. We note that the simple relation $F = ma$ is not directly applicable here because, as fuel and oxidant are being consumed, the mass of the rocket is not constant.

In examining the behaviour of an accelerating rocket particular care is needed in selecting the coordinate axes to which measurements of velocities are referred. We here consider our reference axes fixed to the earth and all velocities must be expressed with respect to these axes. We may not consider