Chapter III

A review of the fundamental formulation of stress, strain, and deflection
Outline

• Introduction
• Assumptions and limitations
• Axial loading
• Torsion of circular shafts
• Beam in bending
• Bending of symmetric beams in two planes
• Thin-walled pressure vessels
• Superposition
• Statically indeterminate problems
• Stress and strain transformations*
• Buckling instability of columns
Introduction

• The basic topics from **mechanics of materials I** include...
  – Direct axial load
  – Shear load
  – Torsion in circular shafts
  – Transverse loading of long, straight, narrow beam

• The purpose of this chapter is to provide a concise review of the fundamental formulation (stress, strain, and deflection)
Assumptions and limitations

- Homogeneous, Isotropic and linear strain-stress relations
- Cross section are exact, and constant or gradually varying in the normal direction.
- The point of load or support connection are at suffer large distance for interested point
- The applied load/ or support connection are perfectly positioned geometrically
- Load are static and applied very gradually
- No initial stress effect or residual stresses
Axial loading

- Axial stresses

\[ \sigma = \pm \frac{P_x}{A} \]

(a) elongation of the prismatic bar

(b) freebody diagram
Axial loading

- Axial strains and deflection
- Hooke’s law 1D

\[ \varepsilon_x = \frac{\sigma_x}{E} \]

\[ \varepsilon_y = \varepsilon_z = -\nu \frac{\sigma_x}{E} \]

\[ \varepsilon_x = \pm \frac{P_x}{AE} \]

\[ \varepsilon_y = \varepsilon_z = \mp \nu \frac{P_x}{AE} \]
Important equations

\[ \varepsilon = \frac{\delta}{L} \] (Geometry of deformation)

\[ \sigma = \frac{P}{A} \] (Equilibrium Condition)

\[ \sigma = E\varepsilon \] (Material behavior)

\[ \delta = \frac{PL}{AE} \] (Deformation)

The product \((AE)\) is called *axial rigidity*
Saint-Venant’s principle
STRESS CONCENTRATIONS

\[ K_t = \frac{\sigma_{\text{max}}}{\sigma_{\text{nom}}} \]
STRESS CONCENTRATION CHARTS

\[ \sigma_{\text{nom}} = \frac{P}{A} = \frac{P}{(D - d) t} \]
Stepped bar with multiple loadings

\[ \delta = \sum_{i=1}^{n} \frac{P_i L_i}{A_i E_i} \]
Flexibility versus stiffness

Flexibility, \( f \)

\[ f = \frac{L}{AE} \]

Stiffness, \( k \)

\[ k = \frac{AE}{L} \]
Non-uniform bars

\[ \delta = \int_0^L \frac{P_x \, dx}{A_x E} \]
Example 3.2-1: The limitation of homogeneous materials
Example 3.2-2: Deformation due to weight of the rod
Example 3.2-3: the displacement of two cables

\[ L_{BC} = 600 \text{ mm} \]

\[ \alpha = 30^\circ \quad \beta = 45^\circ \]

\[ P = 80 \text{ kN} \]
Example of a stepped bar

- Section plane at the change-of-load point
- Determine the internal force in each section
- Determine the deformation in each section

\[ \delta = \sum_{i=1}^{n} \frac{P_i L_i}{A_i E_i} \]

\[ \delta_A = \sum \frac{P_i L_i}{A_i E_i} = \frac{140 \times 10^3 (0.5)}{0.002 (70 \times 10^9)} + \frac{400 \times 10^3 (0.3)}{0.003 (200 \times 10^9)} - \frac{150 \times 10^3 (0.4)}{0.003 (200 \times 10^9)} \]

\[ = (0.5 + 0.2 - 0.1) \times 10^{-3} \text{ m} = 0.6 \text{ mm} \]
Example: Displacement of a three-member device

Given: The rigid bar BC is supported by the 5/8 in diameter (rod AB) and ½ in (rod CD) which under load \( P = 12 \) kip at point E. Each of the rods is made of aluminum alloy 6061 with \( E = 10 \times 10^6 \) psi

Find: The deflection at point E

\[
A_{AB} = \frac{\pi}{4} \left(\frac{5}{8}\right)^2 = 0.307 \text{ in.}^2 \\
A_{CD} = \frac{\pi}{4} \left(\frac{1}{2}\right)^2 = 0.196 \text{ in.}^2
\]

\[
\Sigma M_C = 0: \quad -F_{AB}(24) + 12(16) = 0 \quad F_{AB} = 8 \text{ kips}
\]

\[
\Sigma F_y = 0: \quad -12 + 8 + F_{CD} = 0 \quad F_{CD} = 4 \text{ kips}
\]
Example: Displacement of a three-member device

\[ \delta_B = \frac{F_{AB}L_{AB}}{A_{AB}E} = \frac{8 \times 10^3 \times 25}{(0.307)(10 \times 10^6)} = 65.1 \times 10^{-3} \text{ in.} \]

\[ \delta_C = \frac{F_{CD}L_{CD}}{A_{CD}E} = \frac{4 \times 10^3 \times 25}{(0.196)(10 \times 10^6)} = 51 \times 10^{-3} \text{ in.} \]

\[ \delta_E = \delta_C + (\delta_B - \delta_C) \left(\frac{16}{24}\right) = \left[51 + (14.1) \left(\frac{2}{3}\right)\right] 10^{-3} \]

\[ \delta_E = 60.4 \times 10^{-3} \text{ in} \]
Quiz: Axial loading

As shown in Figure below, a rigid rod ABC is suspended by two wires. The wire AD is made of steel have cross-sectional area $A_s = 0.14 \text{ in}^2$ and elastic modulus $E_s = 30 \times 10^6 \text{ psi}$. For the aluminum-alloy wire CE, cross-section area and elastic modulus $A_a = 0.28 \text{ in}^2$ and $E_a = 10 \times 10^6 \text{ psi}$, respectively. Determine the displacement of point B caused by the load $P = 6 \text{ kips}$.
Torsion of circular shafts

Assumptions

• The plane cross sections perpendicular to the axis of the bar remain plane after the application of a torque: points in a given plane remain in that plane after twisting.
• Furthermore, expansion or contraction of a cross section does not occur, nor does a shortening or lengthening of the bar. Thus all normal strains are zero.
• The material is homogeneous and isotropic.
The maximum shear strain at the outer radius \( c \) is given by:

\[
\gamma_{\text{max}} = \frac{c \phi}{L}
\]

Shear strain at any arbitrary radius \( \rho \) is given by:

\[
\gamma = \frac{\rho \phi}{L}
\]
Torsion formula for circular shafts (C. A. Coulomb, in about 1775)

\[ \tau_{\text{max}} = \frac{Tc}{J} \]

\[ \tau = \frac{T \rho}{J} \]
Important equations

In SI units if $T$ is in N-m, $c$ is in meters and $J$ is in $m^4$, then $\tau$ is in Pa. In English units if $T$ is in in-lb/sec, $c$ is in inch and $J$ is in inch$^4$, then $\tau$ is in psi.

$J$ is called the polar moment of inertia of the entire cross section of a bar. For solid circular and hollow cross sections, $J$ is given by the formulas (on the next slide):
Important formulas for “J“

\[ J_{\text{solid}} = 0.5\pi c^4, \text{ where } c \text{ is the radius of the bar} \]

In terms of the bar diameter, \( J = 0.03125\pi d^4 \)

\[ J_{\text{hollow}} = 0.5\pi(c^4 - b^4), \text{ where } c \text{ and } b \text{ are the inner and outer radii of the bar} \]

In terms of the diameters,
\[ J = 0.03125\pi(d_o^4 - d_i^4) \]
Important formulas for “J“

For thin-walled circular members (i.e., \( r/t \geq 10 \)), an approximate formula for J is given by:

\[
J = 2\pi r^3 t,
\]

where \( r \) is the mean radius and \( t \) is the thickness of the tube, respectively.
Example 3.31: A transmitted torque of step shaft
Example of an application
Coupling bolts
Angle of twist
Equation for angle of twist

Geometry of deformation:
\[ \gamma_{\text{max}} = c\phi/L \]

Equilibrium condition:
\[ \tau_{\text{max}} = Tc/J \]

Material behavior:
\[ \gamma_{\text{max}} = \tau_{\text{max}}/G \]
Equation for angle of twist

Combining the previous equations yield:

$$\phi = \frac{TL}{GJ}$$

Torsional spring stiffness:

$$k = \frac{T}{\phi} = \frac{GJ}{L}$$
Stepped bar with multiple loads

\[ \phi = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i} \]
Torsion failures

(a) Brittle material
(b) Ductile material
Torsion tests
Design of circular shafts

\[ P = \frac{2\pi nT}{60} \]

\[ T = \frac{159P}{f} = \frac{9550P}{n} \]
In SI units:

\[ T = \frac{159P}{f} = \frac{9550P}{n} \]

In English units:

\[ T = \frac{1050P}{f} = \frac{63,000P}{n} \]
Case study
Beam in bending

- Shear Force (SF) & Bending Moment (BM) Equations and Diagram
  - Isolation internal transverse planar surface
  - Singularity functions
- Bending stresses
- Transverse shear stresses
- Bending strain and deflections
- Bending of symmetric beams in two planes
Types of beams

Simple beam

Cantilever beam

Overhang beam
Sign convention

In this text book..

For some other text book..
In this text book...

\[ dV_y + q(x) \, dx = 0 \]

\[ \frac{dV_y}{dx} = -q(x) \]

\[ V_y = -\int q(x) \, dx \]

\[ M_z + dM_z - M_z + V_y \frac{dx}{2} + (V_y + dV_y) \frac{dx}{2} = 0 \]

\[ \frac{dM_z}{dx} + V_y + \frac{1}{2} dV_y = 0 \]

\[ \frac{dM_z}{dx} = -V_y \]
For some other text book..

\[ \frac{dV}{dx} = w \]

\[ V = \int w \, dx \]

= area of load diagram

\[ \frac{dM}{dx} = V \]

\[ M = \int V \, dx \]

= area of shear diagram
<table>
<thead>
<tr>
<th>Loading</th>
<th>Shear Diagram $\frac{dv}{dx} = w$</th>
<th>Moment Diagram $\frac{dm}{dx} = V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td><img src="image" alt="Shear Diagram A" /></td>
<td><img src="image" alt="Moment Diagram A" /></td>
</tr>
<tr>
<td>B</td>
<td><img src="image" alt="Shear Diagram B" /></td>
<td><img src="image" alt="Moment Diagram B" /></td>
</tr>
<tr>
<td>C</td>
<td><img src="image" alt="Shear Diagram C" /></td>
<td><img src="image" alt="Moment Diagram C" /></td>
</tr>
<tr>
<td>D</td>
<td><img src="image" alt="Shear Diagram D" /></td>
<td><img src="image" alt="Moment Diagram D" /></td>
</tr>
<tr>
<td>E</td>
<td><img src="image" alt="Shear Diagram E" /></td>
<td><img src="image" alt="Moment Diagram E" /></td>
</tr>
</tbody>
</table>

- **A**: Positive $V$-jump
- **B**: Zero slope
- **C**: Constant positive slope
- **D**: Positive slope that increases from $V_1$ to $V_2$
- **E**: Positive slope that decreases from $w_1$ to $w_2$
Singularity functions

Forming a single equation for describe any discontinuous functions.

\[ F_n(x) = \langle x - a \rangle^n \]

Where \( n \) is any integer

For \( n < 0 \)
\[ F_n(x) = \langle x - a \rangle^n = \begin{cases} \pm \infty & \text{when } x = a \\ 0 & \text{when } x \neq a \end{cases} \]

For \( n \geq 0 \)
\[ F_n(x) = \langle x - a \rangle^n = \begin{cases} (x-a)^n & \text{when } x > a \\ 0 & \text{when } x \leq a \end{cases} \]
<table>
<thead>
<tr>
<th>Case</th>
<th>Load on Beam</th>
<th>Equivalent Distributed Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1" alt="" /></td>
<td>$w(x) = M_0(x-a)^{-2}$</td>
</tr>
<tr>
<td>2</td>
<td><img src="image2" alt="" /></td>
<td>$w(x) = P(x-a)^{-1}$</td>
</tr>
<tr>
<td>3</td>
<td><img src="image3" alt="" /></td>
<td>$w(x) = w(x-a)^0$</td>
</tr>
<tr>
<td>4</td>
<td><img src="image4" alt="" /></td>
<td>$w(x) = \frac{w_0}{b} (x-a)^1$</td>
</tr>
<tr>
<td>5</td>
<td>( w(x) = \frac{w_0}{b^2} (x-a)^2 )</td>
<td></td>
</tr>
<tr>
<td>---</td>
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<td></td>
</tr>
<tr>
<td>6</td>
<td>( w(x) = w(x-a_1)^0 - w(x-a_2)^0 )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>( w(x) = \frac{w_0}{b} (x-a_1)^1 - \frac{w_0}{b} (x-a_2)^1 - w_0(x-a_2)^0 )</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>( w(x) = w_0(x-a_1)^0 - \frac{w_0}{b} (x-a_1)^1 + \frac{w_0}{b} (x-a_2)^1 )</td>
<td></td>
</tr>
</tbody>
</table>
Example D.2-1: load-intensity equation

\[ q(x) = R_A \langle x-0 \rangle^{-1} - w \langle x-0 \rangle^{0} + w \langle x-a \rangle^{0} - M_C \langle x-b \rangle^{-2} - P_D \langle x-c \rangle^{-1} + R_E \langle x-L \rangle^{-1} \]

\[ q(x) = R_A \langle x \rangle^{-1} - w \langle x \rangle^{0} + w \langle x-a \rangle^{0} - M_C \langle x-b \rangle^{-2} - P_D \langle x-c \rangle^{-1} + R_E \langle x-L \rangle^{-1} \]
Integration rules

\[ \int (x-a)^n \, dx = \begin{cases} (x-a)^{n+1} + c & \text{for } n < 0 \\ \frac{1}{n+1}(x-a)^{n+1} + c & \text{for } n \geq 0 \end{cases} \]

Example

\[ \int (x-a)^{-2} \, dx = (x-a)^{-1} + c \]

\[ \int (x-a)^{1} \, dx = \frac{1}{2}(x-a)^{2} + c \]
Example 3.4-1: SFD and BMD of Overhang beam
Normal stresses

Distribution of bending stress in a beam.

In this formula, $S$ is called the section modulus ($S = I/c$)
Doubly symmetric cross-sectional shapes.

\[ I = \frac{\pi r^4}{4} = \frac{\pi d^4}{64} \]
\[ S = \frac{\pi r^3}{4} = \frac{\pi d^3}{32} \]

\[ I_z = \frac{bh^3}{12} \]
\[ S = \frac{bh^2}{6} \]
Example 3.4-2: Simply supported beam

\[ M_x = \begin{cases} 
600x & 0 < x < 50 \text{ in} \\
-200(7x - 500) & 50 < x < 100 \text{ in} \\
2000(x - 120) & 100 < x < 120 \text{ in} 
\end{cases} \]

\[ M_x = 600x - 2000(x - 50)^1 - 3400(x - 100)^1 - 2000(x - 120)^2 \]
Example:
Single-overhang beam with distributed load and T-cross section
Two-plane bending problem for practice

Calculate the diameter of the shaft based on maximum bending stress
Stress concentration in bending

\[ \sigma_{\text{max}} = K \sigma_{\text{nom}} = K \frac{Mc}{I} \]
\[ \sigma_{\text{nom}} = \frac{Mc}{l} = \frac{6M}{td^2} \]
The diagram shows a graph with the following information:

- $D/d = \infty$ indicates a curve on the graph.
- The formula for nominal stress, $\sigma_{nom} = \frac{Mc}{l} = \frac{6M}{td^2}$, is also shown.
- The graph plots $K$ against $r/d$.

The graph includes a shaded area indicating the region of interest for the nominal stress calculation.
Transverse shear stresses
The shear formula

\[ \tau_{xy} = \frac{V}{Ib} \int_{A^*} y \, dA = \frac{VQ}{Ib} \]

Q is the first moment of area given by

\[ Q = \int_{A^*} y \, dA = A^* \bar{y} \]
Shear Stress Distribution in Rectangular Beams

\[ \tau_{xy} = \frac{V}{2I} \left( \frac{h^2}{4} - y_1^2 \right) \]

\[ \tau_{\text{max}} = \frac{Vh^2}{8I} = \frac{Vh^2}{8bh^3/12} = \frac{3}{2} \frac{V}{A} \]
Example 3.4-3: Shear stress distribution in solid rectangular cross section beam

\[(\tau_{xy})_{\text{max}} = \frac{3V_y}{2A}\]
\[ \tau_{\text{max}} = \frac{VQ}{Ib} = \frac{4V}{3\pi c^2} = \frac{4V}{3A} \]
HOLLOW CIRCULAR SECTION BEAMS

\[ \tau_{\text{max}} = \frac{VQ}{Ib} = \frac{4V}{3A} \frac{c_2^2 + c_2 c_1 + c_1^2}{c_2^2 + c_1^2} \]
Comparison of Shear and Bending Stresses

\[ \tau_{\text{max}} = \frac{3}{2} \frac{V}{A} = \frac{3}{2} \frac{P/2}{bh} = \frac{3}{4} \frac{P}{bh} \]

\[ \sigma_{\text{max}} = \frac{Mc}{I} = \frac{(PL/4)(h/2)}{bh^3/12} = \frac{3}{2} \frac{PL}{bh^2} \]
If $L = 10h$ (“long” beam), then this ratio is only $1/20$, which means that $\tau_{\text{max}}$ is only 5\% of $\sigma_{\text{max}}$. 

\[
\frac{\tau_{\text{max}}}{\sigma_{\text{max}}} = \frac{1}{2} \left( \frac{h}{L} \right)
\]
Bending strain and deflection

\[ \varepsilon_x = -\frac{M_z y}{EI_z} \quad \varepsilon_y = \varepsilon_z = \nu \frac{M_z y}{EI_z} \]

\[ \gamma_{xy} = \frac{2(1+\nu) V_y Q}{E I_z b} \]
Bending strain and deflection

Positive loads and internal force resultants.
\[ \kappa = \frac{1}{\rho} = \frac{d^2v}{dx^2} \ \frac{1 + (dv/dx)^2}{[1 + (dv/dx)^2]^{3/2}} \]

\[ \kappa = \frac{1}{\rho} = \frac{d^2v}{dx^2} \]

\[ \kappa = \frac{1}{\rho} = \frac{M}{EI} \]

\[ \frac{d^2v}{dx^2} = \frac{M}{EI} \]

Bernoulli-Euler Law
Deflection = \( v \)

Slope = \( \theta = \frac{dv}{dx} = v' \)

Moment = \( M = EI \frac{d\theta}{dx} = EIv'' \)

Shear = \( V = \frac{dM}{dx} = (EIv'')' \)

Load = \( w = \frac{dV}{dx} = (EIv'')'' \)
Boundary conditions

1. Fixed support
   - $v(a) = 0$
   - $\theta(a) = 0$

2. Simple support
   - $v(a) = 0$
   - $M(a) = 0$

3. Free end
   - $V(a) = 0$
   - $M(a) = 0$

4. Guided support
   - $\theta(a) = 0$
   - $V(a) = 0$
Method of integration

Find:
(a) Determine the equation of the elastic curve using the double-integration approach.
(b) Derive the equation of the elastic curve using the multiple-integration approach.
(c) Obtain the maximum deflection and the slopes.
The bending moment is given by:

\[ M = \frac{1}{2} wLx - \frac{1}{2} wx^2 \]

Double integration method:

\[ EI \frac{d^2v}{dx^2} = \frac{1}{2} wLx - \frac{1}{2} wx^2 \]
Integrating twice in $x$ gives:

$$EI \frac{dv}{dx} = \frac{1}{4} wLx^2 - \frac{1}{6} wx^3 + C_1$$

$$EIv = \frac{1}{12} wLx^3 - \frac{1}{24} wx^4 + C_1x + C_2$$

Applying the boundary conditions we obtain:

$$C_2 = 0 \text{ and } C_1 = -wL^3/24$$
Slope and deflection are then given by:

\[
\frac{dv}{dx} = - \frac{w}{24EI} \left( L^3 - 6Lx^2 + 4x^3 \right)
\]

\[
v = - \frac{w}{24EI} \left( L^3 x - 2Lx^3 + x^4 \right)
\]
Largest displacements and largest slopes:

\[ v_{\text{max}} = v_C = -\frac{5wL^4}{384EI} = \frac{5wL^4}{384EI} \]

\[ \theta_A = v'(0) = -\frac{wL^3}{24EI} = \frac{wL^3}{24EI} \]

\[ \theta_B = v'(L) = \frac{wL^3}{24EI} \]
Example 3.4-5: Simple supports with intermediate load

Find: Deflection as function of $x$ of the centroidal axis
- Discontinuous equations
- Singularity functions
Bending of symmetric beam in two planes

\[ \sigma_x = \frac{M_y z}{I_t} - \frac{M_z y}{I_z} \]

\[ \sigma_x = -\frac{M_{z'} y'}{I_{z'}} \]

\[ M_{z'} = \sqrt{M_y^2 + M_z^2} \]

\[ I_{z'} = I_y = I_z \]
Example 3.5-1: Pulley shaft system
Thin-walled pressure vessels

If the ratio of the wall thickness \((t)\) to the inner radius \((r)\) is equal or less than about \(1/10\) (or \(r/t \geq 10\)), the vessel is classified as thin-walled.

In fact, in thin-walled vessels, there is often no distinction made between the inside and outside radii because they are nearly equal.
Membrane equation

\[ \sum F_n = 0 : -2\sigma_\theta tr_\phi \Delta \phi \sin \frac{\Delta \theta}{2} - 2\sigma_\phi tr_\theta \Delta \theta \sin \frac{\Delta \phi}{2} + p \cdot r_\theta \Delta \theta \cdot r_\phi \Delta \phi = 0 \]

\[ \frac{\sigma_\theta}{r_\theta} + \frac{\sigma_\phi}{r_\phi} = \frac{P}{t} \]
Cylindrical pressure vessel

- Tangential stress: $\sigma_\theta$
  \[ \sigma_t = \frac{pr}{t} \]

- Axial (longitudinal) stress: $\sigma_z$
  \[ \sigma_a = \frac{pr}{2t} \]
Spherical pressure vessels

Tangential stress due to internal pressure:

\[ \sigma = \frac{pr}{2t} \]
Example: Pressure capacity of a cylindrical vessel

A cylinder of thickness $t = 3$ mm and diameter $d = 1.5$ m is made of steel with yield strength $\sigma_y = 240$ MPA.

**Find:** the internal pressure $p$ that can be carried by the vessel based on a safety factor of $n_s = 2$ on yielding.

\[ \sigma_{all} = \frac{\sigma_y}{n_s} = \frac{240}{2} = 120 \text{ MPA} \]

For circumferential:  
\[ p = \frac{\sigma_{all} t}{r} = \frac{120 \times 10^6 (0.003)}{0.75} = 480 \text{ kPa} \]

For axial or longitudinal:  
\[ p = 2 \frac{\sigma_{all} t}{r} = 960 \text{ kPa} \]

So the gage pressure may not exceed 480 kPa.
Superposition

Works well for beams with small deflections that follow linear Hooke’s law

For the special case when $a = b = L/2$:

$$v_C = (v_C)_w + (v_C)_P$$

$$v_C = \frac{5wL^4}{384EI} + \frac{PL^3}{48EI}$$
Example: Slope and Deflection of a beam with an overhang

\[ \theta_B = \frac{wL^3}{24EI} - \frac{PaL}{3EI} \]

\[ (v_C)_M = \theta_B a = \frac{wL^3 a}{24EI} - \frac{Pa^2 L}{3EI} \]

\[ v_C = \frac{wL^3 a}{24EI} - \frac{Pa^2}{3EI} (L + a) \]
Example 3.7-1: Superposition
Example 3.7-2: Deflection equation (overhang beam)
Example 3.7-3: State of stress (Transverse load+Torque)
Statically Indeterminate Problems

- Step 1: Solve for all possible unknown reactions
- Step 2: Subtract the number for remaining Eq\(^n\) from the number of remaining unknown
  Ex. \((n = 3 - 2) = 1\)
- Step 3: Eliminate \(n\) of unknown reactions (redundant unknown)
- Step 4: Considering redundant unknown as “applied force” - the deflection and/or rotation at the \(n\) point
- Step 5: Using the method of superposition
- Step 6: Solve the simultaneous Eq\(^n\)
- Step 7: Substitute the results of the Step 6 into Step 1
Example 3.8-2: Deflection equation (overhang beam)
APPENDIX C

Beams in Bending
C.1 Cantilever with End Load.

**External and internal reactions**

\[ R_A = -V_y = F \]
\[ M_A = FL \]
\[ M_x = F(x - L) \]

**Deflection**

\[ v_c = \frac{Fx^2}{6EI} (x - 3L) \]
\[ (v_c)_{x=L} = -\frac{FL^3}{3EI} \]

**Slope**

\[ \theta = \frac{dv_c}{dx} = \frac{Fx}{2EI} (x - 2L) \]

C.2 Cantilever with Intermediate Load.

**External and internal reactions**

\[ R_A = -(V_y)_{AB} = F \quad M_A = Fa \]
\[ (M_x)_{AB} = F(x - a) \quad (M_x)_{AB} = (V_y)_{BC} = 0 \]

**Deflection**

\[ (v_c)_{AB} = \frac{Fx^2}{6EI} (x - 3a) \]
\[ (v_c)_{BC} = \frac{Fa^2}{6EI} (a - 3x) \]
\[ (v_c)_{x=L} = \frac{Fa^2}{6EI} (a - 3L) \]

**Slope**

\[ (\theta)_{AB} = \frac{Fx}{2EI} (x - 2a) \]
\[ (\theta)_{BC} = -\frac{Fa^2}{2EI} \]
C.3 Cantilever with Uniform Load.

**External and internal reactions**

\[ R_A = wL \quad M_A = \frac{wL^2}{2} \]

\[ V_y = -w(L - x) \quad M_x = -\frac{w}{2}(L - x)^2 \]

**Deflection**

\[ v_c = \frac{wx^2}{2AEI} (4Lx - x^2 - 6L^2) \]

\[ (v_c)_{x=L} = -\frac{wL^4}{8EI} \]

**Slope**

\[ \theta = \frac{wx}{6EI} (3Lx - x^2 - 3L^2) \]

C.4 Cantilever with Moment Load.

**External and internal reactions**

\[ R_A = V_y = 0 \quad M_A = M_x = M \]

**Deflection**

\[ v_c = \frac{Mx^2}{2EI} \quad (v_c)_{x=L} = \frac{ML^2}{2EI} \]

**Slope**

\[ \theta = \frac{Mx}{EI} \]
C.5 Simple Supports with Intermediate Load.

External and internal reactions

$$R_A = \frac{Fb}{L} \quad R_c = \frac{Fa}{L}$$

$$(V_y)_{AB} = -R_A \quad (V_y)_{BC} = R_B$$

$$(M_c)_{AB} = \frac{Fbx}{L} \quad (M_c)_{BC} = \frac{Fa}{L}(L - x)$$

Deflection

$$(v_c)_{AB} = \frac{Fbx}{6EIL}(x^2 + b^2 - L^2)$$

$$(v_c)_{BC} = \frac{Fa(L - x)}{6EIL}(x^2 + a^2 - 2Lx)$$

$$\theta_{AB} = \frac{Fb}{6EIL}(3x^2 + b^2 - L^2)$$

$$\theta_{BC} = \frac{Fa}{6EIL}(6Lx - 3x^2 - a^2 - 2L^2)$$

C.6 Simple Supports with Uniform Load.

External and internal reactions

$$R_A = R_B = \frac{wL}{2} \quad V_y = -w\left(\frac{L}{2} - x\right)$$

$$M_x = \frac{wx}{2}(L - x)$$

Deflection

$$v_c = \frac{wx}{24EL}(2Lx^2 - x^3 - L^3)$$

$$\frac{(v_c)_{AB}}{L^2} = \frac{5wL^4}{384EI}$$

Slope

$$\theta = \frac{w}{24EI}(6Lx^2 - 4x^3 - L^3)$$
**C.7** Simple Supports with Moment Load.

**External and internal reactions**

\[
R_a = R_c = \frac{M}{L} \quad V_y = -\frac{M}{L}
\]

\[
(M_e)_{AB} = \frac{Mx}{L} \quad (M_e)_{BC} = \frac{M}{L}(x-L)
\]

**Deflection**

\[
(v_e)_{AB} = \frac{Mx}{6EI}(x^2 + 3a^2 - 6aL + 2x^2)
\]

\[
(v_e)_{BC} = \frac{M}{6EI}[x^3 - 3Lx^2 + x(2L^2 + 3a^2)]
\]

**Slope**

\[
(\theta)_{AB} = \frac{M}{6EI} (3x^2 + 3a^2 - 6aL + 2x^2)
\]

\[
(\theta)_{BC} = \frac{M}{6EI} (3x^2 - 6Lx + 2x^2 + 3a^2)
\]

---

**C.8** Simple Supports with Overhanging Load.

**External and internal reactions**

\[
R_A = \frac{Fa}{L} \quad R_B = \frac{F}{L}(L+a)
\]

\[
(V_y)_{AB} = \frac{Fa}{L} \quad (V_y)_{BC} = -F
\]

\[
(M_e)_{AB} = \frac{Faux}{L} \quad (M_e)_{BC} = F(x-L-a)
\]

**Deflection**

\[
(v_e)_{AB} = \frac{Faux}{6EI}(L^2 - x^2)
\]

\[
(v_e)_{BC} = \frac{F(x-L)}{6EI} [(x-L)^2 - a(x-L)]
\]

**Slope**

\[
(\theta)_{AB} = \frac{Fa}{6EI} (L^2 - 3x^2)
\]

\[
(\theta)_{BC} = \frac{F}{6EI} [3x^2 - 6a(L+a) + L(3L+4a)]
\]
**External and internal reactions**

\[ R_A = \frac{Fb}{2L^3} (3L^2 - b^3) \quad R_C = \frac{Fa^2}{2L^3} (3L - a) \]

\[ M_A = \frac{Fb}{2L^3} (L^2 - b^3) \]

\[ (V_y)_{AB} = -R_A \quad (V_y)_{BC} = R_C \]

\[ [M_x]_{AB} = \frac{Fb}{2L^3} [b^2L - L^3 + x(3L^2 - b^3)] \]

\[ (M_x)_{BC} = \frac{Fa^2}{2L^3} (3L^2 - 3Lx - aL + ax) \]

**Deflection**

\[ (v_c)_{AB} = \frac{Fbx^2}{12EI L^5} [3L(b^2 - L^2) + x(3L^2 - b^2)] \]

\[ (v_c)_{BC} = (v_c)_{AB} - \frac{F(x - a)^3}{6EI} \]

**Slope**

\[ (\theta)_{AB} = \frac{Fbx}{4EI L^3} [2L(b^2 - L^2) + x(3L^2 - b^2)] \]

\[ (\theta)_{BC} = (\theta)_{AB} - \frac{F(x - a)^2}{2EI} \]