Chapter 4

Deflection and Stiffness

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Deflection and Stiffness

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A **spring** is a mechanical element that exerts a force when deformed.

If we designate the general relationship between force and deflection by the equation

$$ F = F(y) $$

then **spring rate** is defined as

$$ k(y) = \lim_{\Delta y \to 0} \frac{\Delta F}{\Delta y} = \frac{dF}{dy} $$

where $y$ must be measured in the direction of $F$ and at the point of application of $F$.

For linear force-deflection problems, $k$ is a constant, also called the **spring constant**

$$ k = \frac{F}{y} $$
Tension, Compression, and Torsion

- The total extension or contraction of a uniform bar in pure tension or compression, is given by
  \[ \delta = \frac{Fl}{AE} \]

- The spring constant of an axially loaded bar is then
  \[ k = \frac{AE}{l} \]

- The angular deflection of a uniform round bar subjected to a twisting moment \( T \) is
  \[ \theta = \frac{Tl}{GJ} \]

  where \( \theta \) is in radians

- The torsional spring rate is
  \[ k = \frac{T}{\theta} = \frac{GJ}{l} \]
Deflection Due to Bending

- The curvature of a beam subjected to a bending moment $M$ is given by
  \[
  \frac{1}{\rho} = \frac{M}{EI}
  \]
  where $\rho$ is the radius of curvature

- The slope of the beam at any point $x$ is
  \[
  \theta = \frac{dy}{dx}
  \]

- Therefore
  \[
  \frac{M}{EI} = \frac{d^2y}{dx^2}
  \]
  \[
  \frac{V}{EI} = \frac{d^3y}{dx^3}
  \]
  \[
  \frac{q}{EI} = \frac{d^4y}{dx^4}
  \]
Example 4-1

Determine the equation for the slope and deflection of the beam, the slopes at the ends, and the max deflection.

Solution

The bending moment equation: \[ M = \frac{wl}{2} x - \frac{w}{2}x^2 \quad \text{for} \quad 0 \leq x \leq l \]

Integrating; \[ EIv'' = EI \frac{d^2y}{dx^2} = M \]

Integrating; \[ EIv' = \int M \, dx = \frac{wl}{4}x^2 - \frac{w}{6}x^3 + C_1 \]

Integrating; \[ EIv = \int \int M \, dx = \frac{wl}{12}x^3 - \frac{w}{24}x^4 + C_1x + C_2 \]
Solution

For the boundary condition, we have

\[ x = 0, \ v = 0 \quad \rightarrow \quad C_2 = 0 \]

\[ x = l, \ v = 0 \quad \rightarrow \quad C_1 = -\frac{wl^3}{24} \]

Therefore,

\[ v = \frac{wx}{24EI} (2lx^2 - x^3 - l^3) \]

\[ \theta = v' = \frac{w}{24EI} (6lx^2 - 4x^3 - l^3) \quad \text{Comparing with Table A-9} \]
Beam Deflection Methods

• There are many techniques employed to solve the integration problem for beam deflection. Some of the popular methods include:
  ✓ Superposition (see Sec. 4–5)
  ✓ The moment-area method
  ✓ Singularity functions (see Sec. 4–6)
  ✓ Numerical integration

• Beam Deflections by Superposition:
  Superposition resolves the effect of combined loading on a structure by determining the effects of each load separately and adding the results algebraically.

• Beam Deflections by Singularity Functions:
  Singularity functions are excellent for managing discontinuities, and their application to beam deflection is a simple extension.
Example 4-2
Consider the uniformly loaded beam with a concentrated force. Using superposition, determine the reaction and deflection as a function of $x$.

Solution  
Using superposition method, we have

\[ R_1 = \frac{Fb}{l}, \quad R_2 = \frac{Fa}{l} \]

\[ y_{AB} = \frac{Fbx}{6EI} \left( x^2 + b^2 - l^2 \right) \]

\[ y_{BC} = \frac{Fa(l-x)}{6EI} \left( x^2 + a^2 - 2lx \right) \]

\[ y = \frac{wx}{24EI} \left( 2lx^2 - x^3 - l^3 \right) \]
Solution

Therefore,

\[ R_1 = \frac{F_b}{l} + \frac{wl}{2} \]

\[ R_2 = \frac{F_a}{l} + \frac{wl}{2} \]

\[ y_{AB} = \frac{F_b x}{6EI l} (x^2 + b^2 - l^2) + \frac{wx}{24EI} (2lx^2 - x^3 - l^3) \]

\[ y_{BC} = \frac{F_a (l-x)}{6EI l} (x^2 + a^2 - 2lx^2) + \frac{wx}{24EI} (2lx^2 - x^3 - l^3) \]
Example 4-3
Consider the beam shown in Figure. Determine the deflection equations using superposition.

Solution
For the segment AB, we have

\[ y = \frac{wx}{24EI} (2lx^2 - x^3 - l^3) \]

\[ y_{AB} = \frac{F ax}{6EI l} (l^2 - x^2) \]

\[ y_{BC} = \frac{F(x - l)}{6EI} [(x - l)^2 - a(3x - l)] \]
Solution

Therefore,

\[ y_{AB} = \frac{wx}{24EI} (2lx^2 - x^3 - l^3) + \frac{Fax}{6Ell} (l^2 - x^2) \]

For the segment BC, the deflection is straight since there is no bending moment due to \( w \). The slope of beam at point B is \( \theta_B \) or \( y' \) where \( x = l \)

\[ \theta_B = \frac{dy}{dx} = \frac{d}{dx} \left[ \frac{wx}{24EI} (2lx^2 - x^3 - l^3) \right] \]

\[ \theta_B \bigg|_{x=l} = \frac{w}{24EI} (6ll^2 - 4l^3 - l^3) = \frac{wl^3}{24EI} \]
Solution

The deflection in segment BC due to $w$ is $\theta_b(x-l)$ and adding this to the deflection due to $F$ in BC, yields

$$y_{BC} = \frac{wl^3}{24EI} (x-l) + \frac{F(x-l)}{6EI} \left[ (l-x)^2 - a(3x-l) \right]$$
Strain Energy

• The external work done on an elastic member in deforming it is transformed into strain, or potential, energy.

• The energy is equal to the product of the average force and the deflection, or

$$U = \frac{F}{2} y = \frac{F^2}{2k}$$

• For tension and compression

$$U = \frac{F^2 l}{2AE}$$

or

$$U = \int \frac{F^2}{2AE} dx$$

tension and compression

• The strain energy due to direct shear

$$U = \frac{F^2 l}{2AG}$$

or

$$U = \int \frac{F^2}{2AG} dx$$

direct shear

• The strain energy for torsion is given by

$$U = \frac{T^2 l}{2GJ}$$

or

$$U = \int \frac{T^2}{2GJ} dx$$

torsion
Strain Energy in Beam Bending

- The strain energy stored in a section of the elastic curve of length $ds$ is $dU = (M/2)d\theta$.

$$dU = \frac{M\,ds}{2\rho}$$

- For small deflections, $ds = dx$. Then, for the entire beam

$$U = \int dU = \int \frac{M^2}{2EI} \, dx$$

- Summarized to include both the integral and nonintegral form, the strain energy for bending is

$$U = \begin{cases} 
\frac{M^2l}{2EI} & \text{bending} \\
\int \frac{M^2}{2EI} \, dx & \text{nonintegral}
\end{cases}$$

- The strain energy due to shear loading of a beam can be approximated as

$$U = \begin{cases} 
\frac{CV^2l}{2AG} & \text{transverse shear} \\
\int \frac{CV^2}{2AG} \, dx & \text{nonintegral}
\end{cases}$$

Beam Cross-Sectional Shape | Factor $C$
---|---
Rectangular | 1.2
Circular | 1.11
Thin-walled tubular, round | 2.00
Box sections† | 1.00
Structural sections† | 1.00

†Use area of web only.
Example 4-8

A cantilever beam with a round cross section has a concentrated load $F$ at the free end, as shown in Figure. Find the strain energy in the beam.

**Solution**

Consider the FBD, we have

$\begin{align*}
V &= -F \\
M &= -Fx
\end{align*}$

For the transverse shear, the correction factor $C = 1.11$ (circular cross section).

The transverse shear strain energy is

$$U_{\text{shear}} = \int \frac{CV^2}{2AG} \, dx = \frac{1.11F^2l}{2AG}$$

The bending strain energy is

$$U_{\text{bend}} = \int \frac{M^2}{2EI} \, dx = \frac{1}{2EI} \int_0^l (-Fx)^2 \, dx = \frac{F^2l^3}{6EI}$$

Therefore, the total strain energy is

$$U_{\text{total}} = U_{\text{bend}} + U_{\text{shear}} = \frac{F^2l^3}{6EI} + \frac{1.11F^2l}{2AG}$$
Castigliano’s Theorem

- Castigliano’s theorem states that “when forces act on elastic systems subject to small displacements, the displacement corresponding to any force, in the direction of the force, is equal to the partial derivative of the total strain energy with respect to that force”.

- Mathematically, the theorem of Castigliano is

\[
\delta_i = \frac{\partial U}{\partial F_i}
\]

where \( \delta_i \) is the displacement of the point of application of the force \( F_i \) in the direction of \( F_i \).

- Castigliano’s theorem can be used to find the deflection at a point even though no force or moment acts there.

  ✓ Set up the equation for the total strain energy \( U \)
  ✓ Find an expression for the desired deflection \( \delta \)
  ✓ Since \( Q \) is a fictitious force, solve the expression by setting \( Q \) equal to zero.

\[
\begin{align*}
\delta_i & = \frac{\partial U}{\partial F_i} = \int \frac{1}{AE} \left( F \frac{\partial F}{\partial F_i} \right) \, dx \\
\theta_i & = \frac{\partial U}{\partial M_i} = \int \frac{1}{GJ} \left( T \frac{\partial T}{\partial M_i} \right) \, dx \\
\delta_i & = \frac{\partial U}{\partial F_i} = \int \frac{1}{EI} \left( M \frac{\partial M}{\partial F_i} \right) \, dx
\end{align*}
\]

tension and compression

torsion

bending
Example 4-9

A cantilever beam of previous example, is a carbon steel bar 250 mm long with a 25-mm diameter and is loaded by a force $F = 400$ N.

(a) Find the maximum deflection using Castigliano’s theorem (including that due to shear effect)

(b) What error is introduced if shear is neglected?

Solution

From previous example,

$$U_{total} = \frac{F^2 l^3}{6EI} + \frac{1.11F^2 l}{2AG}$$

According to Castigliano’s theorem, the deflection of the end is

$$\delta_{max} = \frac{\partial U}{\partial F} = \frac{Fl^3}{3EI} + \frac{1.11Fl}{AG}$$

For the area properties,

$$I = \frac{\pi d^4}{64} = \frac{\pi (25)^4}{64} = 19175 \text{ mm}^4$$

$$A = \frac{\pi d^2}{4} = \frac{\pi (25)^2}{4} = 491 \text{ mm}^2$$
Solution

Substituting these values, \( F = 400 \, N, \, l = 0.25 \, m, \, E = 209 \, GPa, \, G = 79 \, GPa \)

\[
\delta_{\text{max}} = 0.52 + 0.003 = 0.523 \, mm
\]

Note that the result is positive because it is in the same direction as the force \( F \).

b) If neglecting shear, we have

\[
\delta_{\text{max}} = \frac{(0.523 - 0.52)}{0.523} = 0.0057 \, mm = 0.57\%
\]
Example 4-10

Using Castigliano’s method, determine the deflections of point A and B due to the force $F$ applied at the end of the step shaft shown in Figure. (The second area moments for section $AB$ and $BC$ are $I_1$ and $2I_1$, respectively)

**Solution**

For $0 \leq x \leq l$, the bending moment is $M = -Fx$

The deflection at $A$ is

$$
\delta_A = \frac{\partial U}{\partial F} = \int_0^l \frac{1}{EI} \left( M \frac{\partial M}{\partial F} \right) \, dx
$$

Substituting,

$$
\delta_A = \frac{1}{E} \left[ \int_0^{l/2} \frac{1}{I_1} (-Fx)(-x) \, dx + \int_{l/2}^l \frac{1}{2I_1} (-Fx)(-x) \, dx \right]
$$

$$
= \frac{1}{E} \left[ \frac{Fl^3}{24I_1} + \frac{7Fl^3}{48I_1} \right] = \frac{3}{16} \frac{Fl^3}{EI_1}
$$
Solution

For \( B \), a fictitious force \( Q \) is necessary at the point. Assuming \( Q \) acts down at \( B \), and \( x \) is as before, the moment equation is

\[
M = -Fx \quad 0 \leq x \leq l/2
\]

\[
M = -Fx - Q\left(x - \frac{l}{2}\right) \quad l/2 \leq x \leq l
\]

Therefore,

\[
\frac{\partial M}{\partial Q} = 0 \quad 0 \leq x \leq l/2
\]

\[
\frac{\partial M}{\partial Q} = -\left(x - \frac{l}{2}\right) \quad l/2 \leq x \leq l
\]

The deflection at \( B \) is

\[
\delta_B = \left[ \int_0^l \frac{1}{EI} \left( M \frac{\partial M}{\partial Q} \right) \right]_{Q=0} dx
\]

\[
= \frac{1}{EI_1} \int_0^{l/2} (-Fx)(0)dx + \frac{1}{E(2I_1)} \int_{l/2}^l (-Fx)\left[-\left(x - \frac{l}{2}\right)\right] dx
\]

\[
\delta_B = \frac{F}{2EI_1} \left( \frac{x^3}{3} - \frac{lx^2}{4} \right) \bigg|_{l/2}^l = \frac{5}{96} \frac{F{l^3}}{EI_1}
\]
Statically Indeterminate Problems

• A system when it is said to be *statically indeterminate* with more unknown support (reaction) forces and/or moments than static equilibrium equations.

• The extra constraint supports are called *redundant supports*.

• A deflection equation is required for each redundant support reaction in order to obtain a solution.

**Procedure**

✓ Choose the redundant reaction(s).
✓ Write the equations of static equilibrium.
✓ Write the deflection equation(s) for the point(s) at the locations of the redundant reaction(s) in terms of the applied loads and the redundant reaction(s)
✓ The equations can now be solved to determine the reactions.
Example 4-14

Determine the reaction of the indeterminate beam 11 of Appendix A-9

Solution

1. Choose $R_2$ at $B$ to be the redundant reaction.
2. Using static equilibrium equations solve for $R_1$ and $M_1$ in terms of $F$ and $R_2$. This results in

$$R_1 = F - R_2 \quad M_1 = \frac{Fl}{2} - R_2l$$  \hspace{1cm} (1)

3. Write the deflection equation for point $B$ in terms of $F$ and $R_2$. Using superposition of beam 1 of Table A–9 with $F = -R_2$, and beam 2 of Table A–9 with $a = l/2$, the deflection of $B$, at $x = l$, is

$$\delta_B = \frac{-R_2l^2}{6EI}(l - 3l) + \frac{F(l/2)^2}{6EI}\left(\frac{l}{2} - 3l\right) = \frac{R_2l^3}{3EI} - \frac{5Fl^3}{48EI} = 0$$  \hspace{1cm} (2)

4. Equation (2) can be solved for $R_2$ directly. This yields

$$R_2 = \frac{5F}{16}$$  \hspace{1cm} (3)

Next, substituting $R_2$ into Eqs. (1) completes the solution, giving

$$R_1 = \frac{11F}{16} \quad M_1 = \frac{3Fl}{16}$$  \hspace{1cm} (4)
Compression Member-General

• The term *column* is applied to all such members except those in which failure would be by simple or pure compression.
• Columns can be categorized then as:
  ✓ Long columns with central loading
  ✓ Intermediate-length columns with central loading
  ✓ Columns with eccentric loading
  ✓ Struts or short columns with eccentric loading
Long Columns with Central Loading

• If the axial force $P$ shown acts along the centroidal axis of the column, simple compression of the member occurs for low values of the force.

• Under certain conditions, when $P$ reaches a specific value, the column becomes unstable and bending develops rapidly.

• The critical force for the pin-ended column is given by

$$ P_{cr} = \frac{\pi^2 EI}{l^2} $$

which is called the Euler column formula.

• Euler Column formula can be extended to apply to other end-conditions by writing

$$ P_{cr} = \frac{C\pi^2 EI}{l^2} $$

where the constant $C$ depends on the end conditions.
Euler Column Formula: General

- Using the relation $I = Ak^2$, where $A$ is the area and $k$ the radius of gyration. Euler Column Equation can be rearranged as

$$\frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2}$$

where $l/k$ is called the slenderness ratio.

- The quantity $P_{cr}/A$ is the critical unit load. It is the load per unit area necessary to place the column in a condition of unstable equilibrium.

- The factor $C$ is called the end-condition constant, and it may have any one of the theoretical values $1/4$, $1$, $2$, and $4$, depending upon the manner in which the load is applied.

<table>
<thead>
<tr>
<th>Column End Conditions</th>
<th>Theoretical Value</th>
<th>Conservative Value</th>
<th>Recommended Value*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed-free</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Rounded-rounded</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Fixed-rounded</td>
<td>2</td>
<td>1</td>
<td>1.2</td>
</tr>
<tr>
<td>Fixed-fixed</td>
<td>4</td>
<td>1</td>
<td>1.2</td>
</tr>
</tbody>
</table>

*To be used only with liberal factors of safety when the column load is accurately known.
In practical engineering applications where defects such as initial crookedness or load eccentricities exit, the Euler equation can only be used for slenderness ratio greater than \((l/k)_1\).

Most designers select point \(T\) such that \(P_{cr}/A = S_y/2\) with corresponding value of \((l/k)_1\) to be

\[
\left(\frac{l}{k}\right)_1 = \left(\frac{2\pi^2 CE}{S_y}\right)^{1/2}
\]
Johnson Formula for Intermediate-Length Column

- For the slenderness ratio that Euler column formula does not apply, the equation developed by J.B. Johnson as the parabolic or J.B. Johnson formula could be used.

\[
\frac{P_{cr}}{A} = a - b \left( \frac{l}{k} \right)^2
\]

- If the parabola is begun at \( S_y \), then \( a = S_y \).

- If point \( T \) is selected as previously noted, then the value of \((l/k)_1\) and the constant \( b \) is found to be

\[
b = \left( \frac{S_y}{2\pi} \right)^2 \frac{1}{CE}
\]

- Upon substituting the known values of \( a \) and \( b \), we obtain the parabolic equation

\[
\frac{P_{cr}}{A} = S_y - \left( \frac{S_y}{2\pi} \frac{l}{k} \right)^2 \frac{1}{CE} \quad \frac{l}{k} \leq \left( \frac{l}{k} \right)_1
\]
Columns with Eccentric Loading

- Load eccentricities or crookedness are likely to occur during manufacture and assembly.

- A column with the line of action of the column forces separated from the centroidal axis of the column by the eccentricity $e$ results in the differential equation

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{Pe}{EI}$$

- The solution of Eq. (a), for the boundary conditions that $y = 0$ at $x = 0$, is

$$y = e \left[ \tan \left( \frac{1}{2} \sqrt{\frac{P}{EI}} \right) \sin \left( \sqrt{\frac{P}{EI}} x \right) + \cos \left( \sqrt{\frac{P}{EI}} x \right) - 1 \right]$$

- The maximum bending moment also occurs at midspan and is

$$M_{\text{max}} = P(e + \delta) = Pe \sec \left( \frac{1}{2} \sqrt{\frac{P}{EI}} \right)$$
Secant Column Formula

- The magnitude of the maximum compressive stress at midspan is found by superposing the axial component and the bending component.

\[ \sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{Mc}{Ak^2} \]

\[ \sigma_c = \frac{P}{A} \left[ 1 + \frac{ec}{k^2} \sec \left( \frac{l}{2k} \sqrt{\frac{P}{EA}} \right) \right] \]

- By imposing the compressive yield strength \( S_{yc} \) as the maximum value of \( \sigma_c \)

\[ \frac{P}{A} = S_{yc} \frac{1 + (ec/k^2) \sec[(l/2k)\sqrt{P/EA}]}{1 + (ec/k^2) \sec[(l/2k)\sqrt{P/EA}]} \]

- The term \( ec/k^2 \) is called the eccentricity ratio.
Example 4-16

Develop specific Euler equation for the sizes of columns having
a) Round cross section
b) Rectangular cross section

Solution

(a) Using $A = \pi d^2/4$ and $k = \sqrt{I/A} = [(\pi d^4/64)/(\pi d^2/4)]^{1/2} = d/4$

$$\frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2} \quad \rightarrow \quad d = \left(\frac{64P_{cr}l^2}{\pi^3 CE}\right)^{1/4}$$

(b) For the rectangular column, we specify a cross section $h \times b$ with the restriction that $h \leq b$. If the end conditions are the same for buckling in both directions, then buckling will occur in the direction of the least thickness. Therefore

$$I = \frac{bh^3}{12} \quad A = bh \quad k^2 = I/A = \frac{h^2}{12}$$

$$\frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2} \quad \rightarrow \quad b = \frac{12P_{cr}l^2}{\pi^2 CEh^3} \quad h \leq b$$
Example 4-17

Specify the diameter of a round column 1.5 m long that is to carry a maximum load estimated to be 22 kN. Use a design factor \( n_d = 4 \) and consider the ends as pinned (rounded). The column material selected has a minimum yield strength of 500 MPa and a modulus of elasticity of 207 GPa.

Solution

We shall design the column for a critical load of

\[
P_{cr} = n_d P = 4(22) = 88 \text{ kN}
\]

From the round column with \( C = 1 \), we have

\[
d = \left( \frac{64P_{cr}l^2}{\pi^3 CE} \right)^{1/4} = \left[ \frac{64(88)(1.5)^2}{\pi^3(1)(207)} \right]^{1/4} \left( \frac{10^3}{10^9} \right)^{1/4} (10^3) = 37.48 \text{ mm}
\]

Therefore, the preferred size is 40 mm
Solution

The slenderness ratio for this size is

\[
\frac{l}{k} = \frac{l}{d/4} = \frac{1.5 \times 10^3}{40/4} = 150
\]

To be sure that this is an Euler column,

\[
\left( \frac{l}{k} \right)_1 = \left( \frac{2\pi^2 CE}{Sy} \right)^{1/2} = \left[ \frac{2\pi^2(1)(207)}{500} \right]^{1/2} \left( \frac{10^9}{10^6} \right)^{1/2} = 90.4
\]

which indicates that it is indeed an Euler column. So select

\[d = 40 \text{ mm}\]
Example 4-19

The hydraulic cylinder shown in Figure 4–22 has an 80-mm bore and is to operate at a pressure of 5.6 MPa. With the clevis mount shown, the piston rod should be sized as a column with both ends rounded for any plane of buckling. The rod is to be made of forged AISI 1030 steel without further heat treatment.

(a) Use a design factor $n_d = 3$ and select a preferred size for the rod diameter if the column length is 1.5 m.

(b) Repeat part (a) but for a column length of 0.5 m.

(c) What factor of safety actually results for each of the cases above?

![Figure 4–22](image-url)
Solution

\[ F = 5.6 \left( \frac{\pi}{4} \right) (80^2) = 28\,149 \text{ N}, \quad S_y = 260 \text{ MPa} \]

\[ P_{cr} = n_d F = 3(28\,149) = 84\,447 \text{ N} \]

(a) Assume Euler with \( C = 1 \)

\[ I = \frac{\pi}{64} d^4 = \frac{P_{cr} l^2}{C \pi^2 E} \quad \Rightarrow \quad d = \left[ \frac{64 P_{cr} l^2}{\pi^3 C E} \right]^{1/4} = \left[ \frac{64(84\,447)(1.5)^2}{\pi^3(1)(207)10^9} \right]^{1/4} = 0.0371 \text{ m} \]

Use \( d = 40 \text{ mm}; \quad k = d/4 = 10 \)

\[ \frac{l}{k} = \frac{1500}{10} = 150 \]

\[ \left( \frac{l}{k} \right)_1 = \left( \frac{2\pi^2(1)(207)10^9}{260(10)^6} \right)^{1/2} = 125.4 \quad \therefore \text{ use Euler} \]

\[ P_{cr} = \frac{\pi^2(207)10^9(\pi/64)(0.04^4)}{1.5^2} = 114.1 \text{ kN} \]

\( d = 40 \text{ mm} \) is satisfactory.
Solution

(b) \[ d = \left[ \frac{64 \times (84447)(0.5)^2}{\pi^3(1)(207)10^9} \right]^{1/4} = 0.0214 \text{ m, } \text{so use } 22 \text{ mm} \]

\[ k = \frac{22}{4} = 5.5 \text{ mm} \]

\[ \frac{l}{k} = \frac{500}{5.5} = 90.9 \text{ try Johnson} \]

\[ P_{cr} = \frac{\pi}{4} (0.022^2) \left[ 260(10^6) - \left( \frac{260(10^6)}{2\pi} \right)^2 \frac{1}{1(207)(10^9)} \right] = 98831 \text{ N} \]

Use \( d = 22 \text{ mm} \)

(c) \[ n_{(a)} = \frac{114100}{28149} = 4.05 \]

\[ n_{(b)} = \frac{98831}{28149} = 3.51 \]
Struts or Short Compression Members

- A strut is a short compression member such that the maximum compressive stress in the $x$ direction at point $B$ in an intermediate section is the sum of a simple component $P/A$ and a flexural component $Mc/I$

$$\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{Pe}{IA} = \frac{P}{A} \left( 1 + \frac{ec}{k^2} \right)$$

where $k = (I/A)^{1/2}$ and is the radius of gyration, $c$ is the coordinate of point $B$, and $e$ is the eccentricity of loading.

- How long is a short member?

- If we decide that the limiting percentage is to be 1 percent of $e$, then, from Eq. (4–44), the limiting slenderness ratio turns out to be

$$\left( \frac{l}{k} \right)_2 = 0.282 \left( \frac{AE}{P} \right)^{1/2}$$
Example 4-20

Figure 4–24a shows a workpiece clamped to a milling machine table by a bolt tightened to a tension of 8.9 kN. The clamp contact is offset from the centroidal axis of the strut by a distance \( e = 2.5 \) mm, as shown in part \( b \) of the figure. The strut, or block, is steel, 25 mm square and 0.1 m long, as shown. Determine the maximum compressive stress in the block.

Figure 4–24
Solution

First we find $A = bh = 0.025(0.025) = 625 \times 10^{-6}$ m$^2$, $I = bh^3/12 = 0.025(0.025)^3/12 = 32.55 \times 10^{-9}$ m$^4$, $k^2 = I/A = 32.55 \times 10^{-9}/625 \times 10^{-6} = 52.1 \times 10^{-6}$ m$^2$, and $l/k = 0.1/(52.1 \times 10^{-6})^{1/2} = 13.9$. Equation (4–56) gives the limiting slenderness ratio as

$$\left(\frac{l}{k}\right)_2 = 0.282 \left(\frac{AE}{P}\right)^{1/2} = 0.282 \left[\frac{625 \times 10^{-6} (210 \times 10^9)}{8900}\right]^{1/2} = 34.2$$

Thus the block could be as long as

$$l = 34.2k = 34.2(52.1 \times 10^{-6})^{1/2} = 0.24$$ m

before it need be treated by using the secant formula. So Eq. (4–55) applies and the maximum compressive stress is

$$\sigma_c = \frac{P}{A} \left(1 + \frac{ec}{k^2}\right) = \frac{8900}{625 \times 10^{-6}} \left[1 + \frac{0.0025(0.0125)}{52.1 \times 10^{-6}}\right] = 22.8$$ MPa
1) Using superposition, find the deflection of the steel shaft at A in the figure. Find the deflection at midspan. By what percentage do these two values differ?
2) Using Castigliano’s theorem, determine the maximum deflection for the uniformly loaded cantilever beam 3 (Table A-9). Neglect shear.

3 Cantilever—uniform load

\[
R_1 = wl \quad M_1 = \frac{wl^2}{2}
\]

\[
V = w(l - x) \quad M = -\frac{w}{2}(l - x)^2
\]

\[
y = \frac{wx^2}{24EI} (4lx - x^2 - 6l^2)
\]

\[
y_{\text{max}} = -\frac{wl^4}{8EI}
\]
3) The figure shows a rectangular member OB, made from 6-mm-thick aluminum plate, pinned to the ground at one end and supported by a 12-mm-diameter round steel rod with hooks formed on the ends. A load of 400 N is applied as shown. Determine the vertical deflection at point B by,

a) Using superposition
b) Using Castigliano’s theorem
4) For the beam shown, determine the support reaction using superposition.
Solution 3a)

\[ \Sigma M_O = 0 = 150 F_{AC} - 275(400) \quad \Rightarrow \quad F_{AC} = 733.33 \text{ N} \]

The deflection at point A in the negative y direction is equal to the elongation of the rod AC. From Table A-5, \( E_s = 30 \text{ Mpsi} \).

\[
y_A = -\left( \frac{FL}{AE} \right)_{AC} = -\frac{733.33(0.3)}{\left[ \pi \left(0.012^2/4\right) \times 207 \times (10^9) \right]} = -9.4 \times (10^{-6}) \text{ m} = 0.0094 \text{ mm} 
\]

By similar triangles the deflection at B due to the elongation of the rod AC is

\[
\frac{y_A}{150} = \frac{y_{B1}}{450} \quad \Rightarrow \quad y_{B1} = 3 y_A = 3(-0.0094) = -0.0282 \text{ mm} 
\]

From Table A-5, \( E_a = 71.7 \text{ GPa} \)

The bar can then be treated as a simply supported beam with an overhang AB. From Table A-9, beam 10
Solution 3a)

\[
y_{B_2} = \left(\overline{BD}\right) \left(\frac{dy_{BC}}{dx} \bigg|_{x=l+a}\right) - \frac{Fa^2}{3EI} (l+a) = 0.175 \left\{ \frac{d}{dx} \left( \frac{F(x-l)}{6EI} \left[ (x-l)^2 - a(3x-l) \right] \right) \right\} \bigg|_{x=l+a} - \frac{Fa^2}{3EI} (l+a)
\]

\[
= 0.175 \frac{F}{6EI} \left[ 3(x-l)^2 - 3a(x-l) - a(3x-l) \right] \bigg|_{x=l+a} - \frac{Fa^2}{3EI} (l+a) = - \frac{0.175Fa}{6EI} (2l+3a) - \frac{Fa^2}{3EI} (l+a)
\]

\[
= -\frac{0.175(400)(0.125)}{6[71.7(10^9)(0.006(0.05^3)/12)]} \left[ 2(0.15) + 3(0.125) \right] - \frac{400(0.125^2)}{3[71.7(10^9)(0.006(0.05^3)/12)]} (0.15 + 0.125)
\]

\[
= -0.0003475 \text{ m}
\]

\[
y_B = y_{B_1} + y_{B_2} = -0.0000282 - 0.0003475 = -0.0003757 \text{ m} = -0.376 \text{ mm} \quad \text{Ans.}
\]
Solution 3b)

\[ I_{OB} = \frac{6(50^3)}{12} = 62500 \text{ mm}^4 \]

\[ A_{AC} = \pi \left( \frac{12^2}{4} \right) = 113 \text{ mm}^2 \]

\[ \Sigma M_O = 0 = 0.15R_C - 0.275(400) - 0.45Q \]

\[ R_C = 3Q + 733.3 \]

\[ \Sigma M_A = 0 = 0.15R_O - 0.125(400) - 0.3Q \quad \Rightarrow \quad R_O = 2Q + 333.3 \]

**Bending in OB.**

**BD:** Bending in BD is only due to Q which when set to zero after differentiation gives no contribution.

**AD:** Using the variable \( \bar{x} \) as shown in the figure above

\[ M = -400\bar{x} - Q(0.175 + \bar{x}) \quad \frac{\partial M}{\partial Q} = -(0.175 + \bar{x}) \]

**OA:** Using the variable \( x \) as shown in the figure above

\[ M = -(2Q + 333.3)x \quad \frac{\partial M}{\partial Q} = -2x \]

**Axial in AC:**

\[ F = 3Q + 733.3 \quad \frac{\partial F}{\partial Q} = 3 \]
Solution 3b)

\[
\delta_B = \left( \frac{\partial U}{\partial Q} \right)_{Q=0} = \left[ \left( \frac{FL}{AE} \right) \frac{\partial F}{\partial Q} \right]_{Q=0} + \left( \frac{1}{EI} \sum M \frac{\partial M}{\partial Q} \, dx \right)_{Q=0}
\]

\[
= \frac{733.3(0.3)}{0.000113(207)10^9}(3) + \frac{1}{EI} \int_0^{0.125} (400\bar{x})(0.175 + \bar{x}) \, d\bar{x} + \int_0^{0.15} 2(333.3)x^2 \, d\bar{x}
\]

\[
= 28214(10^{-9}) + \frac{1}{71.7(10^9)62 \cdot 500(10^{-12})} \left[ 400 \int_0^{0.125} \bar{x}(0.175 + \bar{x}) \, d\bar{x} + 666.6 \int_0^{0.15} x^2 \, d\bar{x} \right]
\]

\[
= 28214(10^{-9}) + 2231.52(10^{-7})[400(0.002018) + 666.6(0.001125)]
\]

\[
= 0.000376 \, m = 0.38 \, mm
\]
Solution 4)

4-98  Procedure 1.
1. Choose \( R_B \) as redundant reaction.

2. Statics. \( R_C = wl - R_B \)  \hspace{1cm} (1)

\[
M_C = \frac{1}{2} \omega l^2 - R_B (l - a) \hspace{1cm} (2)
\]

3. Deflection equation for point B. Superposition of beams 2 and 3 of Table A-9,

\[
y_B = \frac{R_B (l - a)^3}{3EI} + \frac{\omega (l - a)^2}{24EI} \left[ 4l (l - a) - (l - a)^2 - 6l^2 \right] = 0
\]

4. Solving for \( R_B \).

\[
R_B = \frac{\omega}{8(l - a)} \left[ 6l^2 - 4l(l - a) + (l - a)^2 \right]
\]

\[
= \frac{\omega}{8(l - a)} \left( 3l^2 + 2al + a^2 \right) \hspace{1cm} \text{Ans.}
\]

Substituting this into Eqs. (1) and (2) gives

\[
R_C = wl - R_B = \frac{\omega}{8(l - a)} \left( 5l^2 - 10al - a^2 \right)
\]

\[
M_C = \frac{1}{2} \omega l^2 - R_B (l - a) = \frac{\omega}{8} \left( l^2 - 2al - a^2 \right)
\]