Tenth Edition

CHAPTER

19

VECTOR MECHANICS FOR ENGINEERS: DYNAMICS

Ferdinand P. Beer
E. Russell Johnston, Jr.
Phillip J. Cornwell
Lecture Notes:
Brian P. Self
California Polytechnic State University
Sanjeev Sanghi
Indian Institute of Technology, Delhi

Mechanical Vibrations

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Electrical Analogues

Because running in the International Space Station might cause unwanted vibrations, they have installed a Treadmill Vibration Isolation System.

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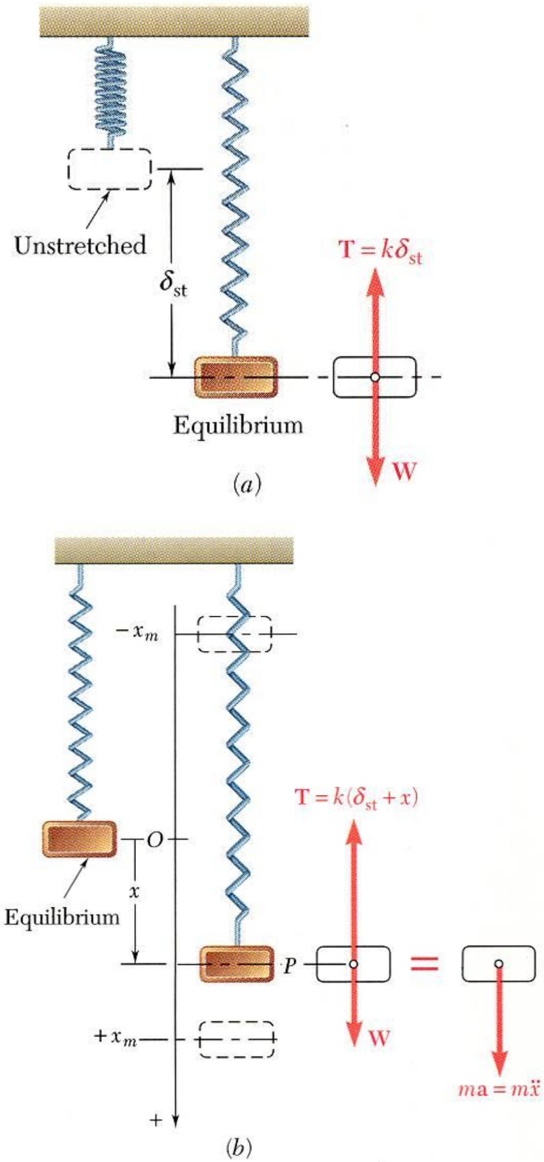
Introduction

- *Mechanical vibration* is the motion of a particle or body which oscillates about a position of equilibrium. Most vibrations in machines and structures are undesirable due to increased stresses and energy losses.
- Time interval required for a system to complete a full cycle of the motion is the *period* of the vibration.
- Number of cycles per unit time defines the *frequency* of the vibrations.
- Maximum displacement of the system from the equilibrium position is the *amplitude* of the vibration.
- When the motion is maintained by the restoring forces only, the vibration is described as *free vibration*. When a periodic force is applied to the system, the motion is described as *forced vibration*.
- When the frictional dissipation of energy is neglected, the motion is said to be *undamped*. Actually, all vibrations are *damped* to some degree.

Free Vibrations of Particles. Simple Harmonic Motion



- If a particle is displaced through a distance x_m from its equilibrium position and released with no velocity, the particle will undergo *simple harmonic motion*,

$$ma = F = W - k(\delta_{st} + x) = -kx$$

$$m\ddot{x} + kx = 0$$

- General solution is the sum of two *particular solutions*,

$$x = C_1 \sin\left(\sqrt{\frac{k}{m}}\,t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$= C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t)$$

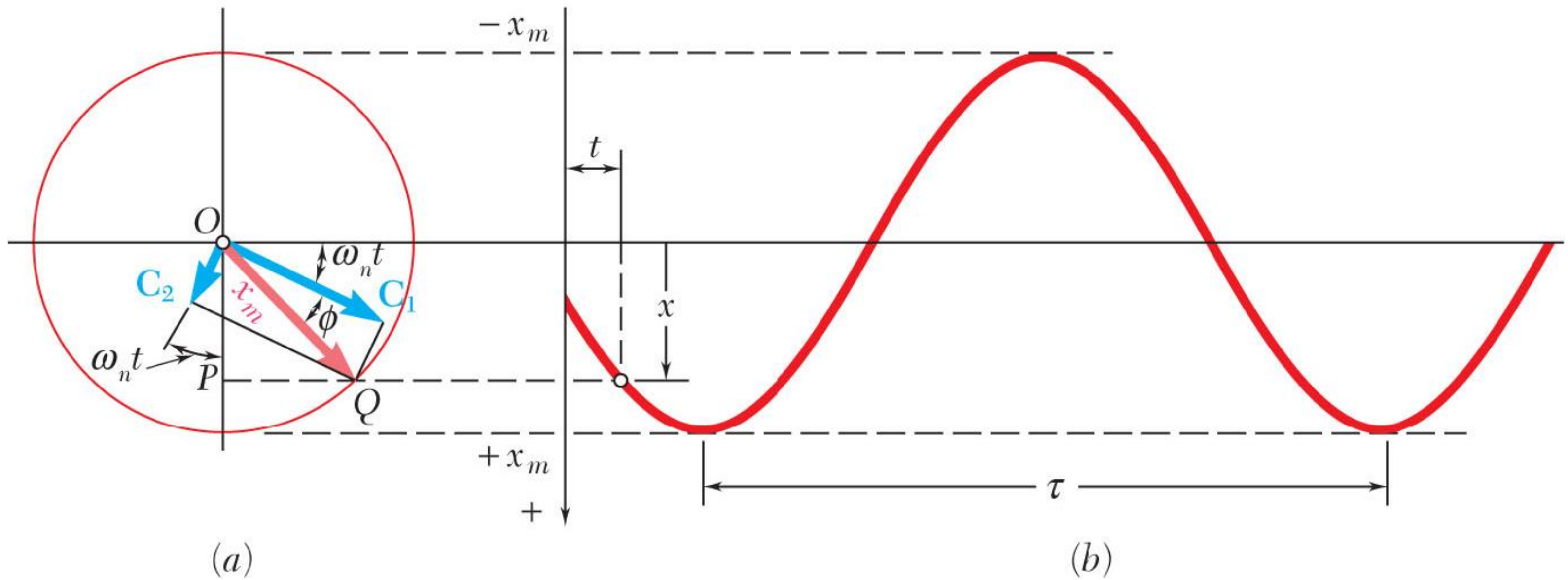
- x is a *periodic function* and ω_n is the *natural circular frequency* of the motion.

- C_1 and C_2 are determined by the initial conditions:

$$x = C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t) \qquad C_2 = x_0$$

$$v = \dot{x} = C_1\omega_n \cos(\omega_n t) - C_2\omega_n \sin(\omega_n t) \qquad C_1 = v_0/\omega_n$$

Free Vibrations of Particles. Simple Harmonic Motion



$$x = x_m \sin(\omega_n t + \phi)$$

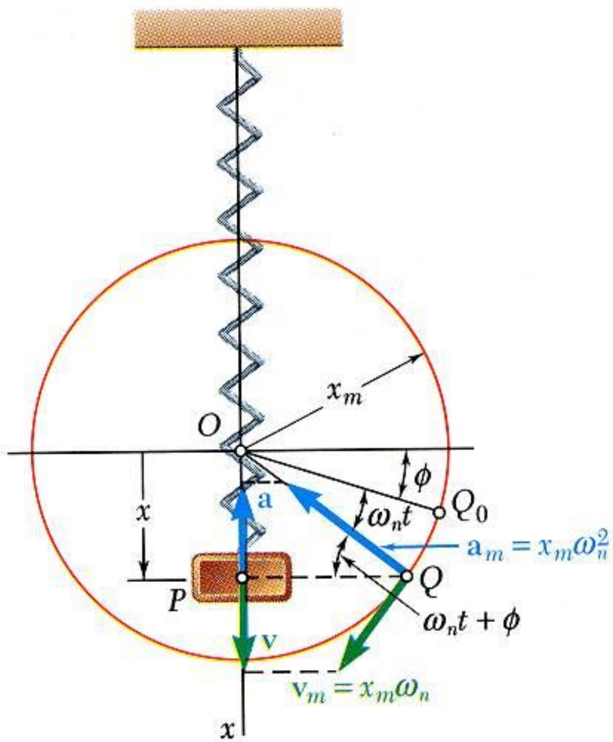
$$x_m = \sqrt{(v_0/\omega_n)^2 + x_0^2} = \textit{amplitude}$$

$$\phi = \tan^{-1}(v_0/x_0\omega_n) = \textit{phase angle}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \textit{period}$$

$$f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi} = \textit{natural frequency}$$

Free Vibrations of Particles. Simple Harmonic Motion



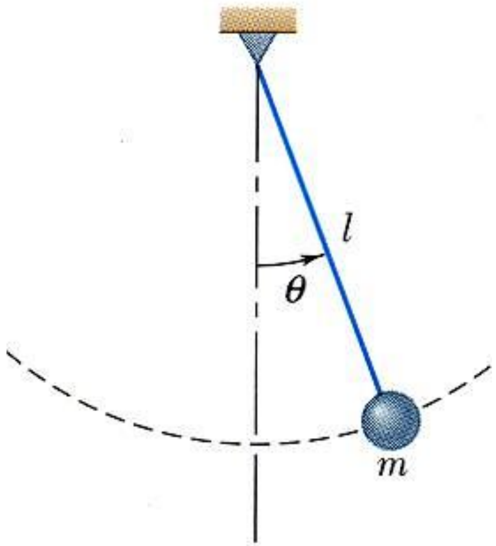
- Velocity-time and acceleration-time curves can be represented by sine curves of the same period as the displacement-time curve but different phase angles.

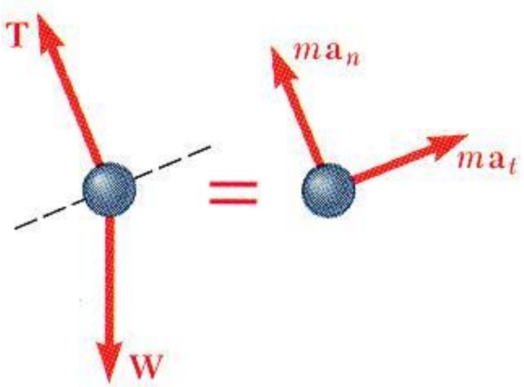
$$x = x_m \sin(\omega_n t + \phi)$$

$$\begin{aligned} v &= \dot{x} \\ &= x_m \omega_n \cos(\omega_n t + \phi) \\ &= x_m \omega_n \sin(\omega_n t + \phi + \pi/2) \end{aligned}$$

$$\begin{aligned} a &= \ddot{x} \\ &= -x_m \omega_n^2 \sin(\omega_n t + \phi) \\ &= x_m \omega_n^2 \sin(\omega_n t + \phi + \pi) \end{aligned}$$

Simple Pendulum (Approximate Solution)





- Results obtained for the spring-mass system can be applied whenever the resultant force on a particle is proportional to the displacement and directed towards the equilibrium position.
- Consider tangential components of acceleration and force for a simple pendulum,

$$\sum F_t = ma_t : \quad -W\sin\theta = ml\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

for small angles,

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\theta = \theta_m \sin(\omega_n t + \phi)$$

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}}$$

Simple Pendulum (Exact Solution)

An exact solution for $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

leads to

$$\tau_n = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\sin^2(\theta_m/2)\sin^2\phi}} \qquad (1)$$

which requires numerical solution.

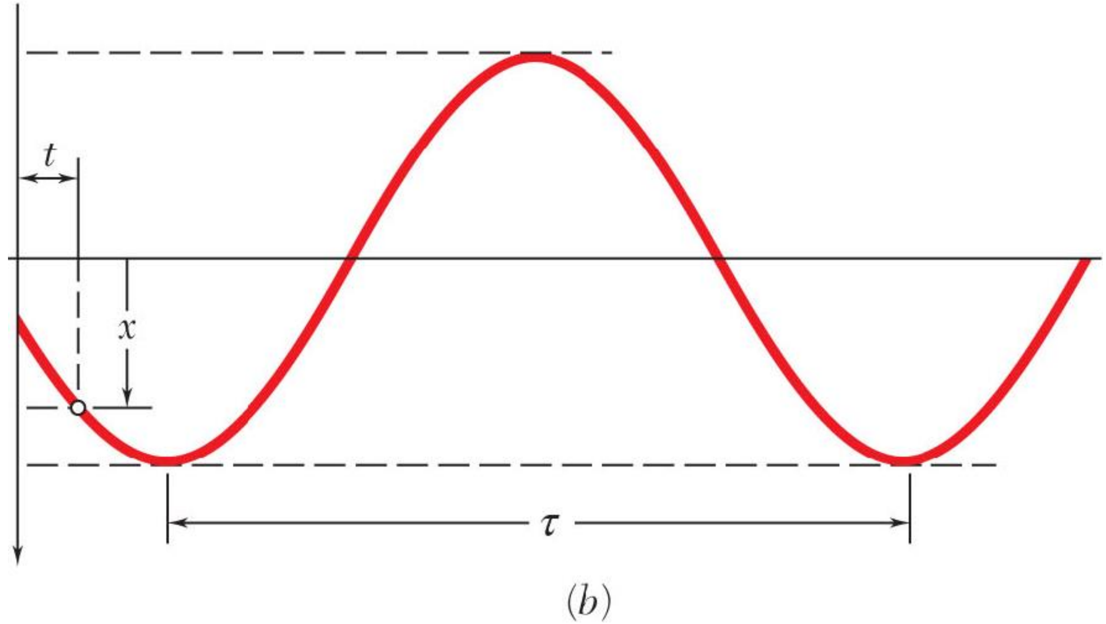
$$\tau_n = \frac{2K}{\pi}\left(2\pi\sqrt{\frac{l}{g}}\right), \text{ with } K \text{ being integral in (1)}$$

TABLE 19.1 Correction Factor for the Period of a Simple Pendulum

θ_m	0°	10°	20°	30°	60°	90°	120°	150°	180°
K	1.571	1.574	1.583	1.598	1.686	1.854	2.157	2.768	∞
$2K/\pi$	1.000	1.002	1.008	1.017	1.073	1.180	1.373	1.762	∞

Concept Question

The amplitude of a vibrating system is shown to the right. Which of the following statements is true (choose one)?



(b)

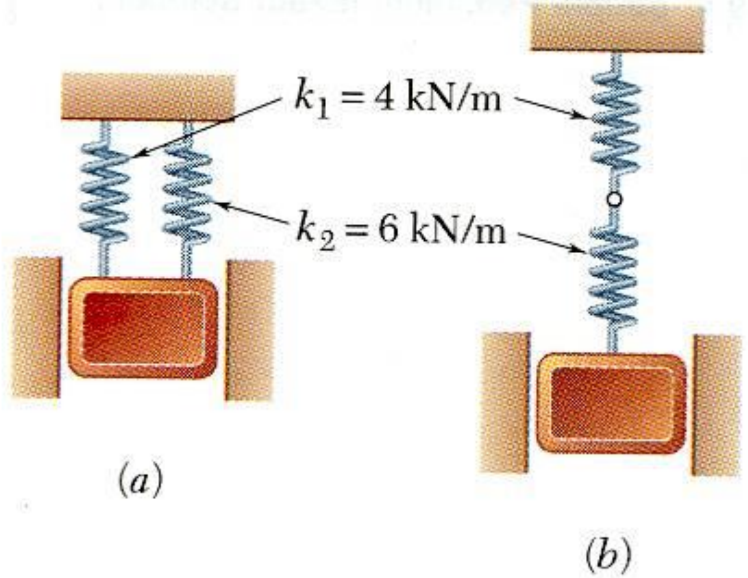
a) The amplitude of the acceleration equals the amplitude of the displacement

b) The amplitude of the velocity is always opposite (negative to) the amplitude of the displacement

c) The maximum displacement occurs when the acceleration amplitude is a minimum

d) The phase angle of the vibration shown is zero

Sample Problem 19.1



A 50-kg block moves between vertical guides as shown. The block is pulled 40mm down from its equilibrium position and released.

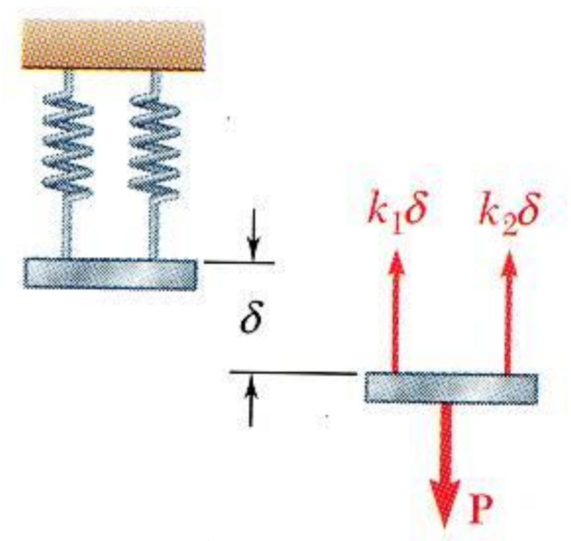
For each spring arrangement, determine *a*) the period of the vibration, *b*) the maximum velocity of the block, and *c*) the maximum acceleration of the block.

<u>SOLUTION</u>:

- For each spring arrangement, determine the spring constant for a single equivalent spring.
- Apply the approximate relations for the harmonic motion of a spring-mass system.

Sample Problem 19.1

$k_1 = 4\,\text{kN/m} \quad k_2 = 6\,\text{kN/m}$



$$P = k_1\delta + k_2\delta$$

$$k = \frac{P}{\delta} = k_1 + k_2$$

$$= 10\,\text{kN/m} = 10^4\,\text{N/m}$$

<u>SOLUTION</u>:

- Springs in parallel:
  - determine the spring constant for equivalent spring
  - apply the approximate relations for the harmonic motion of a spring-mass system

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^4\,\text{N/m}}{20\,\text{kg}}} = 14.14\,\text{rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\boxed{\tau_n = 0.444\,\text{s}}$$

$$v_m = x_m \omega_n$$
$$= (0.040\,\text{m})(14.14\,\text{rad/s})$$

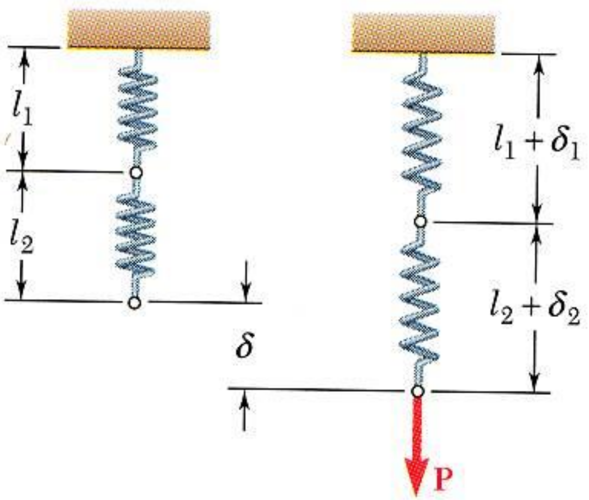
$$\boxed{v_m = 0.566\,\text{m/s}}$$

$$a_m = x_m a_n^2$$
$$= (0.040\,\text{m})(14.14\,\text{rad/s})^2$$

$$\boxed{a_m = 8.00\,\text{m/s}^2}$$

Sample Problem 19.1

$k_1 = 4\,\text{kN/m} \quad k_2 = 6\,\text{kN/m}$



$$d = d_1 + d_2 = \frac{P}{k_1} + \frac{P}{k_2}$$

$$d = \frac{(k_1 + k_2)P}{k_1 k_2}$$

$$k = \frac{P}{d}\frac{k_1 k_2}{k_1 + k_2} = 2.4\,\text{kN/m} = 2400\ \text{N/m}$$

- Springs in series:
  - determine the spring constant for equivalent spring
  - apply the approximate relations for the harmonic motion of a spring-mass system

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2400\text{N/m}}{20\,\text{kg}}} = 6.93\,\text{rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\boxed{\tau_n = 0.907\ \text{s}}$$

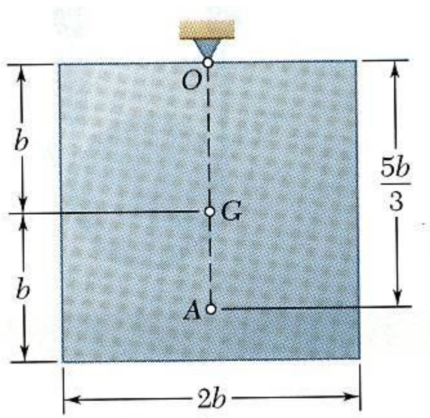
$$v_m = x_m \omega_n$$
$$= (0.040\ \text{m})(6.93\,\text{rad/s})$$

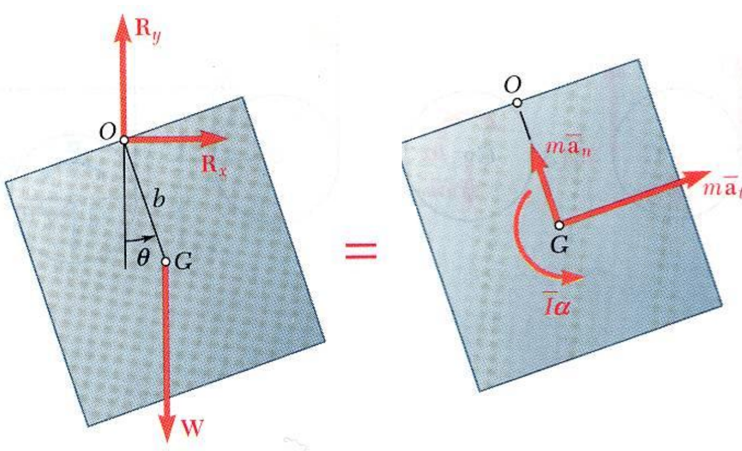
$$\boxed{v_m = 0.277\,\text{m/s}}$$

$$a_m = x_m a_n^2$$
$$= (0.040\ \text{m})(6.93\,\text{rad/s})^2$$

$$\boxed{a_m = 1.920\,\text{m/s}^2}$$

Free Vibrations of Rigid Bodies





- If an equation of motion takes the form

$$\ddot{x} + \omega_n^2 x = 0 \quad \text{or} \quad \ddot{\theta} + \omega_n^2 \theta = 0$$

the corresponding motion may be considered as simple harmonic motion.

- Analysis objective is to determine ω_n.

- Consider the oscillations of a square plate

$$+\circlearrowleft \; -W(b \sin\theta) = (mb\ddot{\theta}) + \bar{I}\ddot{\theta}$$

but $\bar{I} = \frac{1}{12} m\left[(2b)^2 + (2b)^2\right] = \frac{2}{3} mb^2, \quad W = mg$

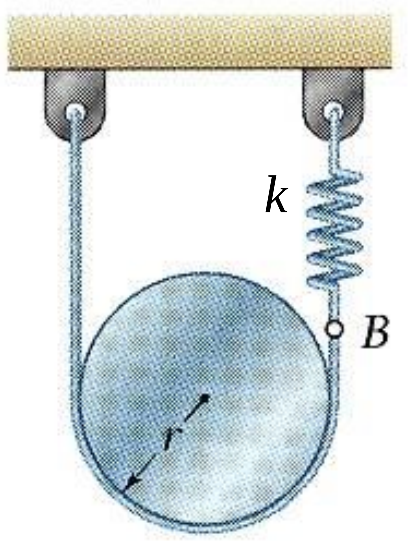
$$\ddot{\theta} + \frac{3}{5}\frac{g}{b}\sin\theta \cong \ddot{\theta} + \frac{3}{5}\frac{g}{b}\theta = 0$$

then $\omega_n = \sqrt{\frac{3g}{5b}}, \quad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{5b}{3g}}$

- For an equivalent simple pendulum,

$$l = 5b/3$$

Sample Problem 19.2



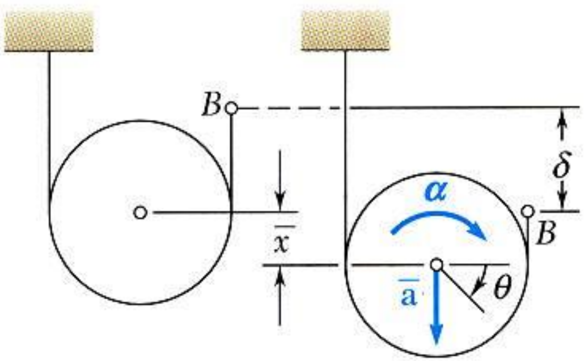
A cylinder of weight *W* is suspended as shown.

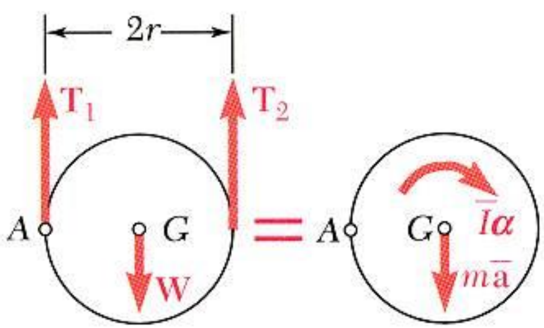
Determine the period and natural frequency of vibrations of the cylinder.

SOLUTION:

- From the kinematics of the system, relate the linear displacement and acceleration to the rotation of the cylinder.
- Based on a free-body-diagram equation for the equivalence of the external and effective forces, write the equation of motion.
- Substitute the kinematic relations to arrive at an equation involving only the angular displacement and acceleration.

Sample Problem 19.2





SOLUTION:

- From the kinematics of the system, relate the linear displacement and acceleration to the rotation of the cylinder.

$$\bar{x} = r\theta \qquad \delta = 2\bar{x} = 2r\theta$$

$$\vec{\alpha} = \ddot{\theta} \; +\!\circlearrowright \qquad \bar{a} = r\alpha = r\ddot{\theta} \qquad \vec{\bar{a}} = r\ddot{\theta} \; +\!\downarrow$$

- Based on a free-body-diagram equation for the equivalence of the external and effective forces, write the equation of motion.

$$+\!\circlearrowright \sum M_A = \sum (M_A)_{eff} : \qquad Wr - T_2(2r) = m\bar{a}r + \bar{I}\alpha$$

but $T_2 = T_0 + k\delta = \frac{1}{2}W + k(2r\theta)$

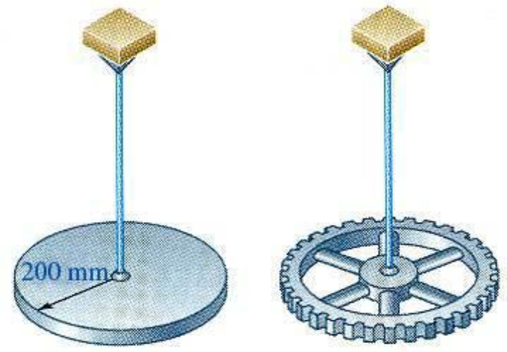
- Substitute the kinematic relations to arrive at an equation involving only the angular displacement and acceleration.

$$Wr - \left(\tfrac{1}{2}W + 2kr\theta\right)(2r) = m\left(r\ddot{\theta}\right)r + \tfrac{1}{2}mr^2\theta$$

$$\ddot{\theta} + \frac{8}{3}\frac{k}{m}\theta = 0$$

$$\omega_n = \sqrt{\frac{8k}{3m}} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3m}{8k}} \qquad f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{8k}{3m}}$$

Sample Problem 19.3



$m = 10\ \text{kg}$

$\tau_n = 1.13\ \text{s}$ $\qquad$ $\tau_n = 1.93\ \text{s}$

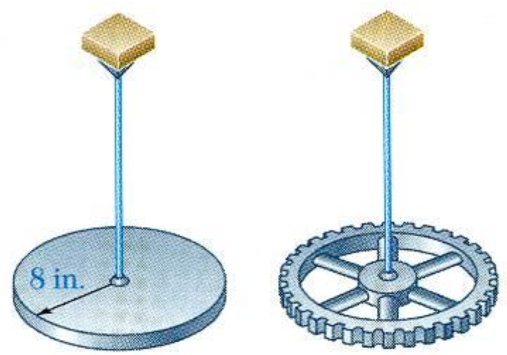
The disk and gear undergo torsional vibration with the periods shown. Assume that the moment exerted by the wire is proportional to the twist angle.

Determine *a*) the wire torsional spring constant, *b*) the centroidal moment of inertia of the gear, and *c*) the maximum angular velocity of the gear if rotated through 90° and released.

<u>SOLUTION</u>:

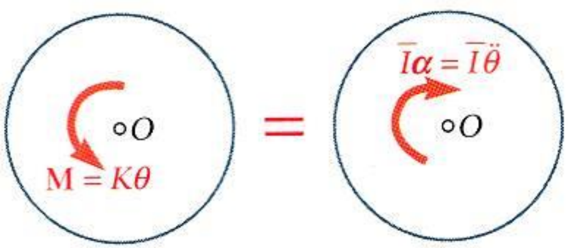
- Using the free-body-diagram equation for the equivalence of the external and effective moments, write the equation of motion for the disk/gear and wire.
- With the natural frequency and moment of inertia for the disk known, calculate the torsional spring constant.
- With natural frequency and spring constant known, calculate the moment of inertia for the gear.
- Apply the relations for simple harmonic motion to calculate the maximum gear velocity.

Sample Problem 19.3



$W = 20\ \text{lb}$

$\tau_n = 1.13\,\text{s} \qquad \tau_n = 1.93\ \text{s}$



SOLUTION:

- Using the free-body-diagram equation for the equivalence of the external and effective moments, write the equation of motion for the disk/gear and wire.

$$+\circlearrowleft \sum M_O = \sum (M_O)_{eff} : \qquad +K\theta = -\bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$

$$\omega_n = \sqrt{\frac{K}{\bar{I}}} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\bar{I}}{K}}$$

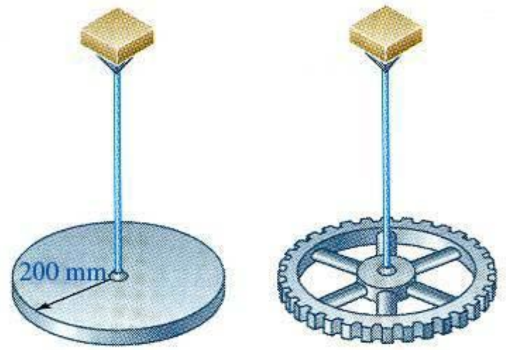
- With the natural frequency and moment of inertia for the disk known, calculate the torsional spring constant.

$$\bar{I} = \tfrac{1}{2}mr^2 = \frac{1}{2}(10\ \text{kg})(0.2\ \text{m})^2 = 0.2\ \text{kg}\cdot\text{m}^2$$

$$1.13 = 2\pi\sqrt{\frac{0.2}{K}}$$

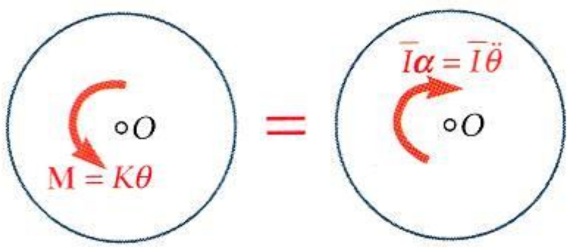
$$\boxed{K = 6.18\ \text{N}\cdot\text{m/rad}}$$

Sample Problem 19.3



$W = 20\text{ lb}$

$\tau_n = 1.13\text{s}$ $\qquad$ $\tau_n = 1.93\text{ s}$



$$\omega_n = \sqrt{\frac{K}{\bar{I}}} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\bar{I}}{K}}$$

$$K = 6.18\text{ N}\times\text{m/rad}$$

- With natural frequency and spring constant known, calculate the moment of inertia for the gear.

$$1.93 = 2p\sqrt{\frac{\bar{I}}{6.183}}$$

$$\boxed{\bar{I}_{gear} = 0.583\text{ kg}\times\text{m}^2}$$

- Apply the relations for simple harmonic motion to calculate the maximum gear velocity.

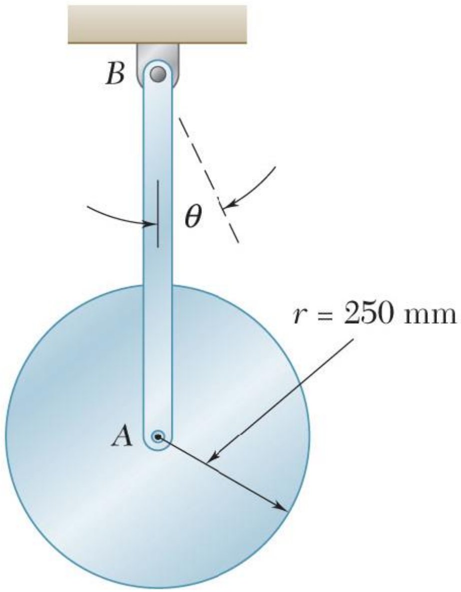
$$\theta = \theta_m \sin\omega_n t \qquad \omega = \theta_m\omega_n \sin\omega_n t \qquad \omega_m = \theta_m\omega_n$$

$$\theta_m = 90° = 1.571\text{ rad}$$

$$\omega_m = \theta_m\left(\frac{2\pi}{\tau_n}\right) = (1.571\text{ rad})\left(\frac{2\pi}{1.93\text{ s}}\right)$$

$$\boxed{\omega_m = 5.11\text{rad/s}}$$

Group Problem Solving



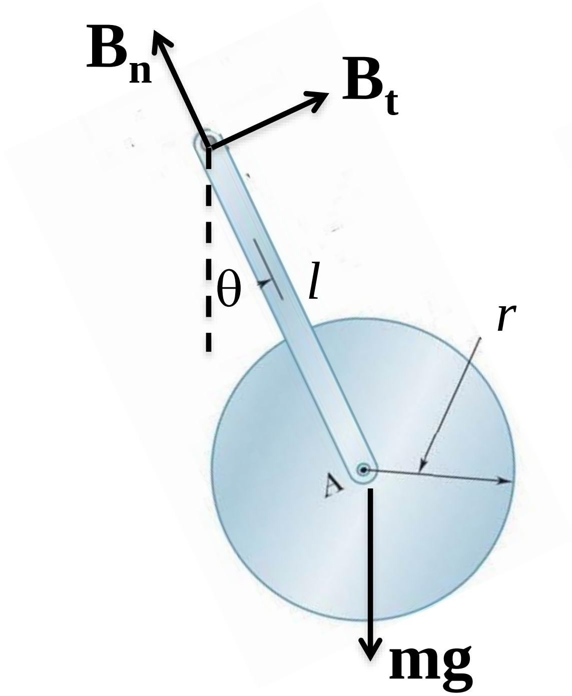
A uniform disk of radius 250 mm is attached at *A* to a 650-mm rod *AB* of negligible mass which can rotate freely in a vertical plane about *B*. If the rod is displaced 2° from the position shown and released, determine the period of the resulting oscillation.

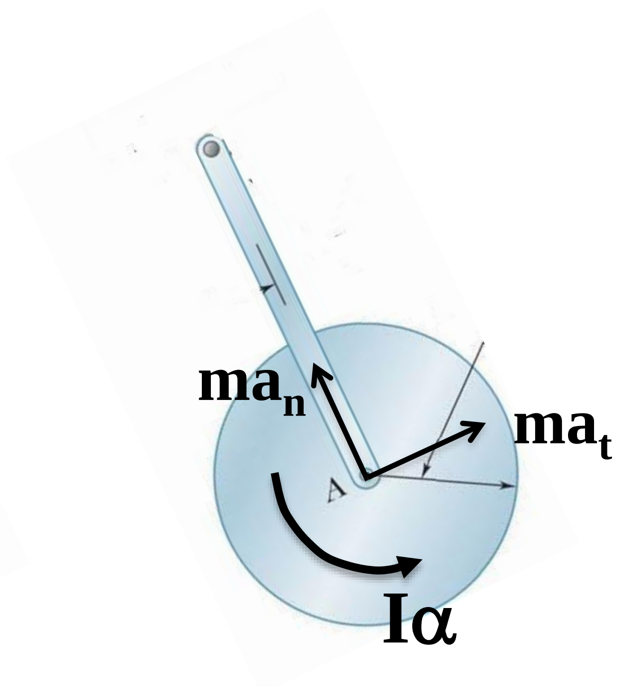
<u>SOLUTION</u>:

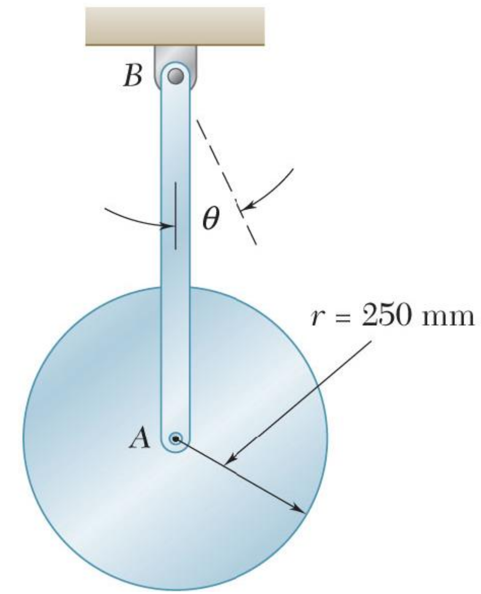
- Using the free-body and kinetic diagrams, write the equation of motion for the pendulum.
- Determine the natural frequency and moment of inertia for the disk (use the small angle approximation).
- Calculate the period.

Group Problem Solving

Draw the FBD and KD of the pendulum (m_{bar} ~ 0).







Determine the equation of motion.

$$\Sigma M_B = I_B \alpha$$

$$-mgl\sin\theta = \left(\bar{I} + ml^2\right)\alpha$$

*Note that you could also do this by using the "moment" from a_t, and that $a_t = l\alpha$

$$-mgl\sin\theta = \bar{I}\alpha + lma_t$$

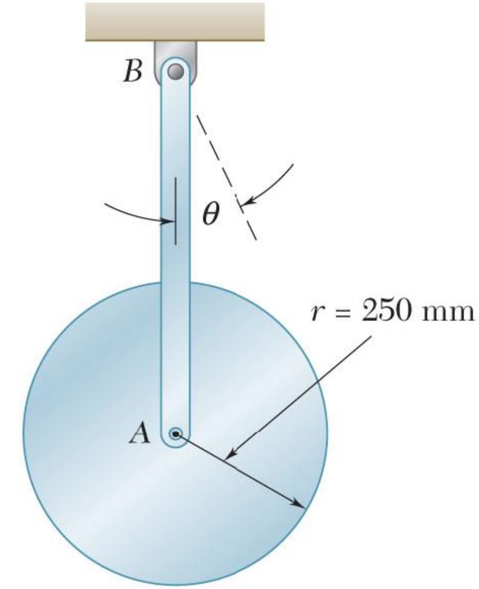
Group Problem Solving

Find I, set up equation of motion using small angle approximation

$$-mgl\sin\theta = \left(\bar{I} + ml^2\right)\alpha$$

$$\bar{I} = \frac{1}{2}mr^2, \quad \sin\theta \approx \theta$$

$$\left(\frac{1}{2}mr^2 + ml^2\right)\ddot{\theta} + mgl\theta = 0$$



Determine the natural frequency

$$\omega_n^2 = \frac{gl}{\left(\frac{r^2}{2} + l^2\right)}$$

$$= \frac{(9.81)(0.650)}{\frac{1}{2}(0.250)^2 + (0.650)^2}$$

$$= 14.053$$

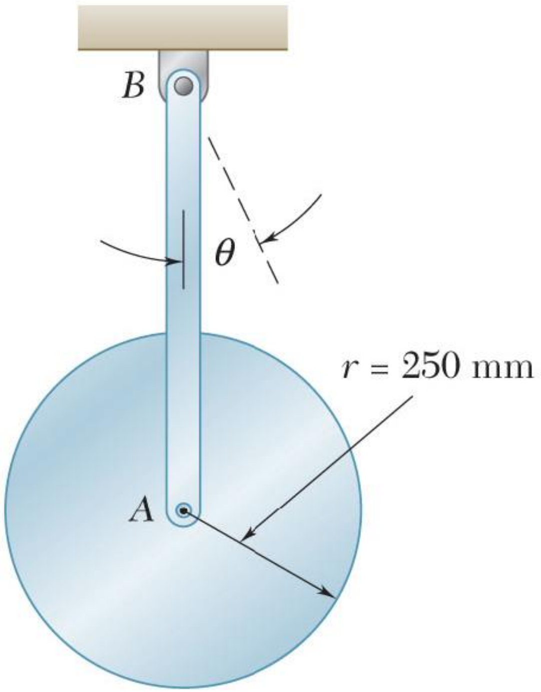
$$\omega_n = 3.7487 \text{ rad/s}$$

Calculate the period

$$\tau_n = \frac{2\pi}{\omega_n} = 1.676 \text{ s}$$

$$\boxed{\tau_n = 1.676 \text{ s}}$$

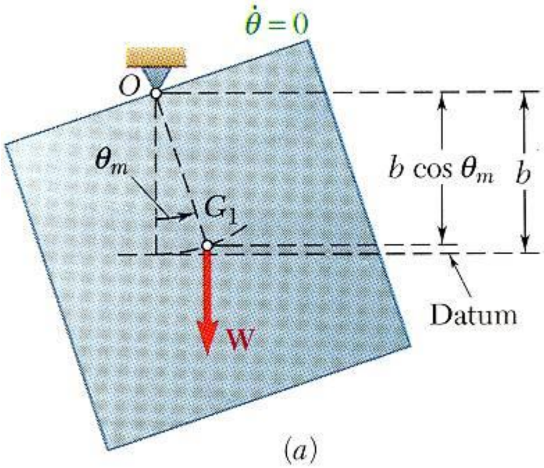
Concept Question



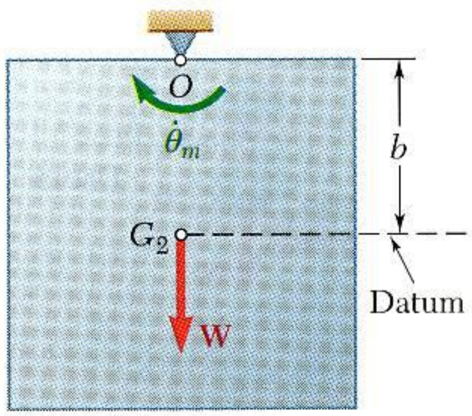
In the previous problem, what would be true if the bar was hinged at A instead of welded at A (choose one)?

a) **The natural frequency of the oscillation would be larger**
b) **The natural frequency of the oscillation would be larger**
c) **The natural frequencies of the two systems would be the same**

Principle of Conservation of Energy



(a)



(b)

- Resultant force on a mass in simple harmonic motion is conservative - total energy is conserved.

$$T + V = \text{constant} \qquad \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} kx^2 = \text{constant}$$

$$\dot{x}^2 + \omega_n^2 x^2 =$$

- Consider simple harmonic motion of the square plate,

$$T_1 = 0 \qquad V_1 = Wb(1-\cos\theta) = Wb\left[2\sin^2(\theta_m/2)\right]$$

$$\cong \tfrac{1}{2} Wb\,\theta_m^2$$

$$T_2 = \tfrac{1}{2} m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2 \qquad V_2 = 0$$

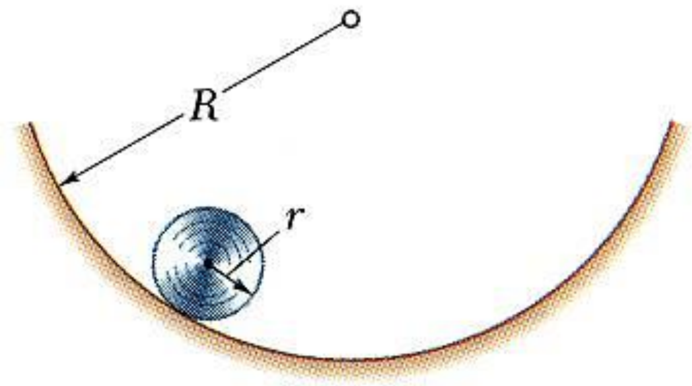
$$= \tfrac{1}{2} m\left(b\dot\theta_m\right)^2 + \tfrac{1}{2}\left(\tfrac{2}{3}mb^2\right)\omega_m^2$$

$$= \tfrac{1}{2}\left(\tfrac{5}{3}mb^2\right)\dot\theta_m^2$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + \tfrac{1}{2} Wb\,\theta_m^2 = \tfrac{1}{2}\left(\tfrac{5}{3}mb^2\right)\theta_m^2\omega_n^2 + 0 \qquad \omega_n = \sqrt{3g/5b}$$

Sample Problem 19.4

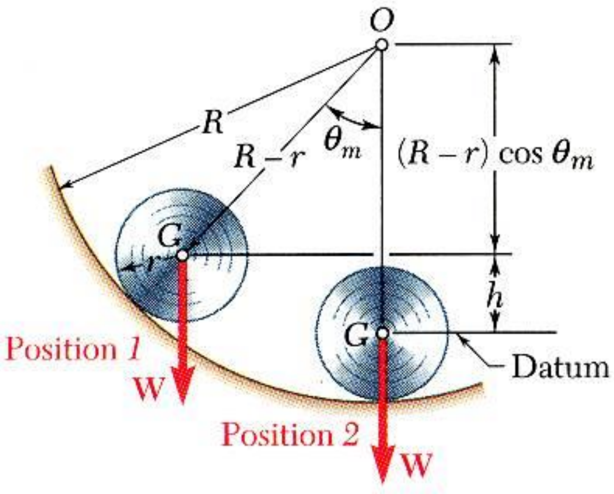


Determine the period of small oscillations of a cylinder which rolls without slipping inside a curved surface.

SOLUTION:

- Apply the principle of conservation of energy between the positions of maximum and minimum potential energy.
- Solve the energy equation for the natural frequency of the oscillations.

Sample Problem 19.4



SOLUTION:

- Apply the principle of conservation of energy between the positions of maximum and minimum potential energy.

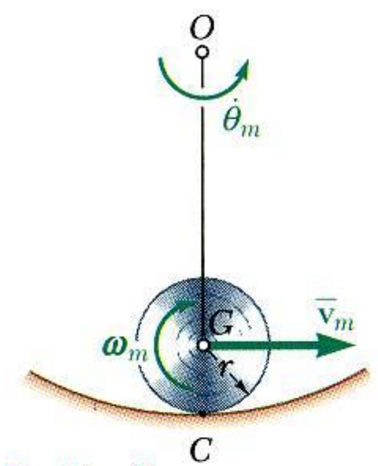
$$T_1 + V_1 = T_2 + V_2$$

$$T_1 = 0 \qquad V_1 = Wh = W(R-r)(1-\cos\theta) \cong W(R-r)\left(\theta_m^2/2\right)$$

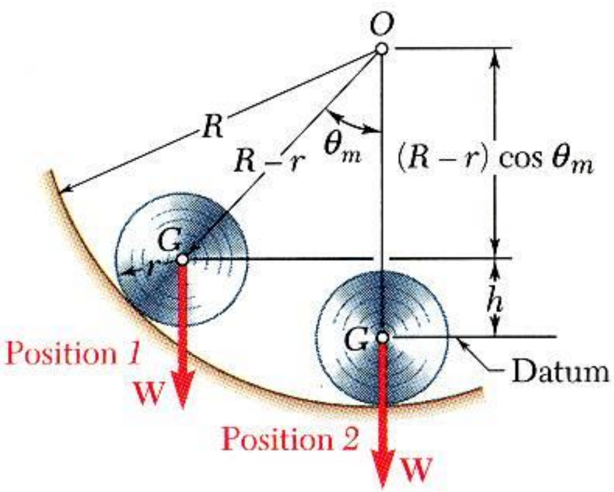
$$T_2 = \tfrac{1}{2} m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2 \qquad V_2 = 0$$

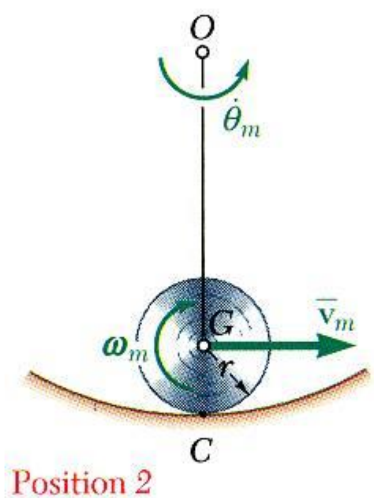
$$= \tfrac{1}{2} m(R-r)\dot{\theta}_m^2 + \tfrac{1}{2}\left(\tfrac{1}{2} mr^2\right)\left(\frac{R-r}{r}\right)^2 \dot{\theta}_m^2$$

$$= \tfrac{3}{4} m(R-r)^2 \dot{\theta}_m^2$$



Sample Problem 19.4





- Solve the energy equation for the natural frequency of the oscillations.

$$T_1 = 0 \qquad V_1 \cong W(R-r)\left(\theta_m^2/2\right)$$

$$T_2 = \tfrac{3}{4}m(R-r)^2\dot{\theta}_m^2 \qquad V_2 = 0$$

$$T_1 + V_1 = T_2 + V_2$$

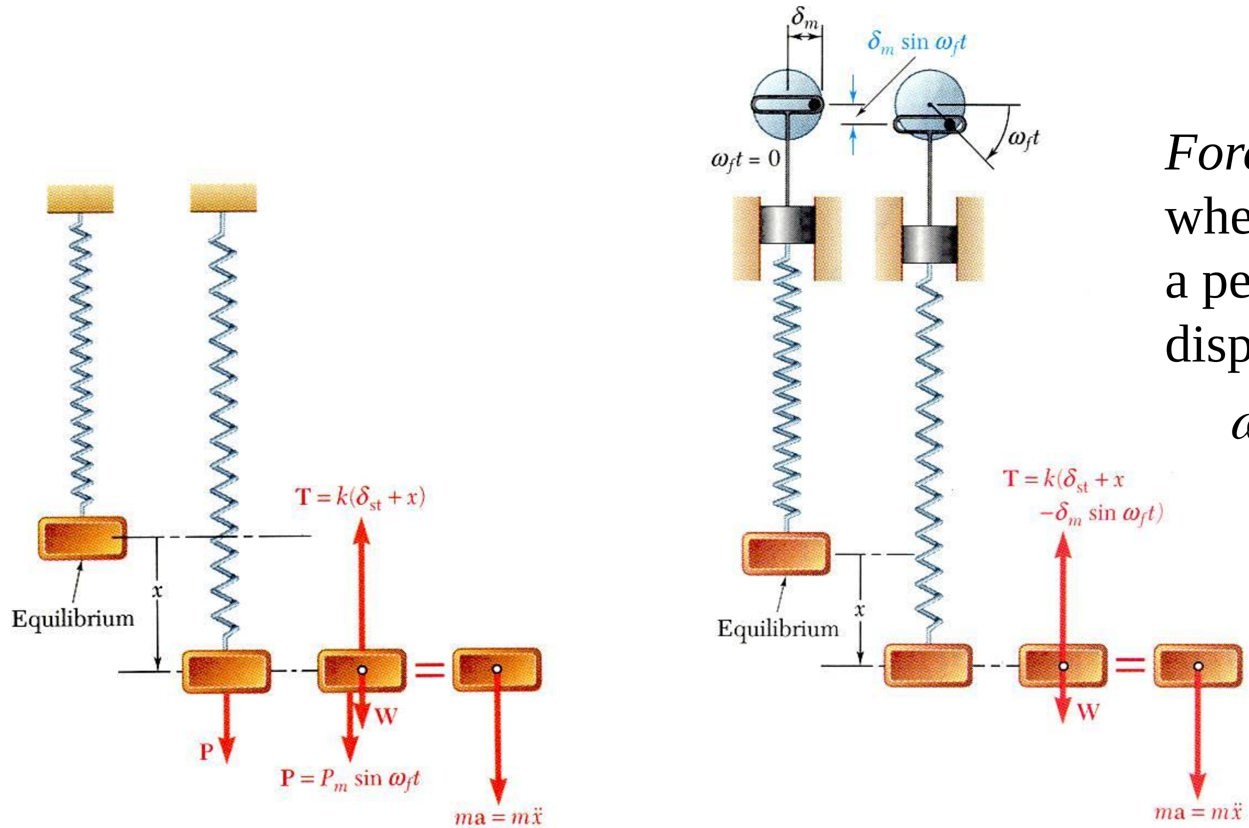
$$0 + W(R-r)\frac{\theta_m^2}{2} = \tfrac{3}{4}m(R-r)^2\dot{\theta}_m^2 + 0$$

$$(mg)(R-r)\frac{\theta_m^2}{2} = \tfrac{3}{4}m(R-r)^2\left(\theta_m\omega_n\right)_m^2$$

$$\omega_n^2 = \frac{2}{3}\frac{g}{R-r}$$

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3}{2}\frac{R-r}{g}}$$

Forced Vibrations



Forced vibrations - Occur when a system is subjected to a periodic force or a periodic displacement of a support.

$$\omega_f = \textit{forced frequency}$$

$$+\downarrow \sum F = ma:$$

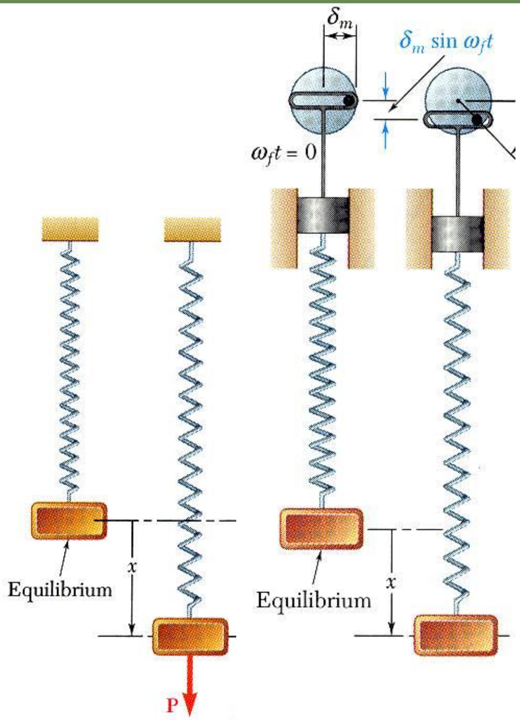
$$P_m \sin \omega_f t + W - k(\delta_{st} + x) = m\ddot{x}$$

$$m\ddot{x} + kx = P_m \sin \omega_f t$$

$$W - k(\delta_{st} + x - \delta_m \sin \omega_f t) = m\ddot{x}$$

$$m\ddot{x} + kx = k\delta_m \sin \omega_f t$$

Forced Vibrations



$$m\ddot{x} + kx = P_m \sin \omega_f t$$

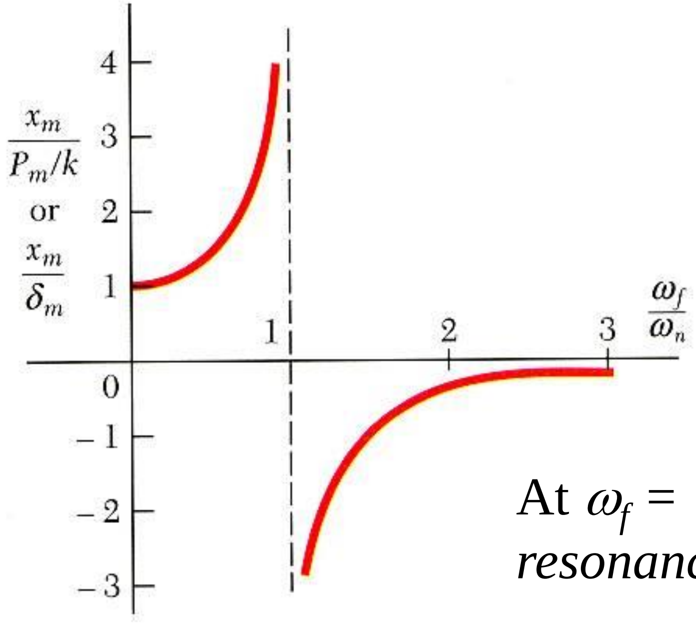
$$m\ddot{x} + kx = k\delta_m \sin \omega_f t$$

$$x = x_{complementary} + x_{particular}$$
$$= \left[C_1 \sin \omega_n t + C_2 \cos \omega_n t\right] + x_m \sin \omega_f t$$

Substituting particular solution into governing equation,

$$-m\omega_f^2 x_m \sin \omega_f t + kx_m \sin \omega_f t = P_m \sin \omega_f t$$

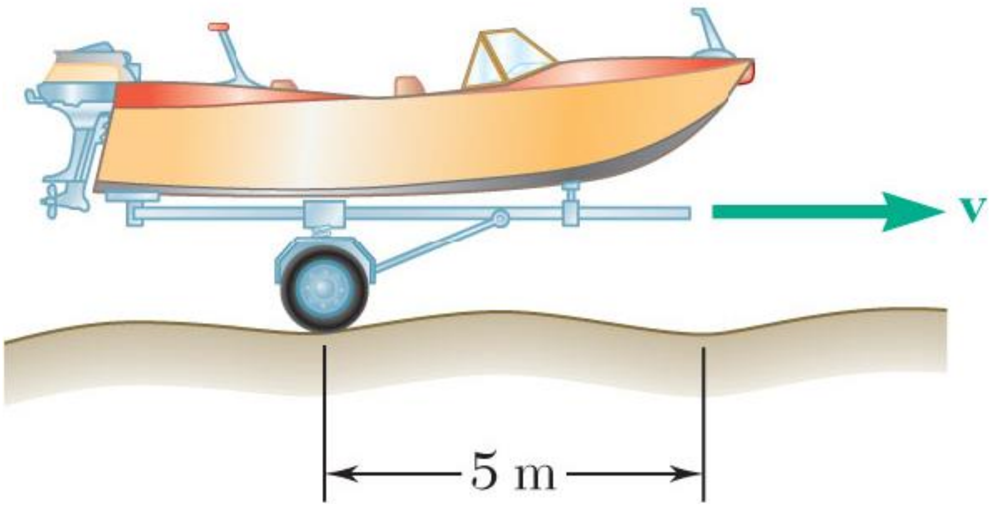
$$x_m = \frac{P_m}{k - m\omega_f^2} = \frac{P_m / k}{1 - (\omega_f / \omega_n)^2} = \frac{\delta_m}{1 - (\omega_f / \omega_n)^2}$$



At $\omega_f = \omega_n$, forcing input is in *resonance* with the system.

Concept Question

A small trailer and its load have a total mass m. The trailer can be modeled as a spring with constant k. It is pulled over a road, the surface of which can be approximated by a sine curve with an amplitude of 40 mm and a wavelength of 5 m. Maximum vibration amplitude occur at 35 km/hr. What happens if the driver speeds up to 50 km/hr?

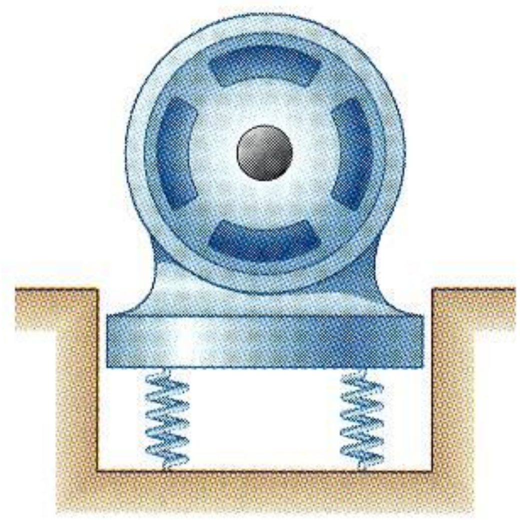


a) The vibration amplitude remains the same.

b) The vibration amplitude would increase.

c) The vibration amplitude would decrease.

Sample Problem 19.5



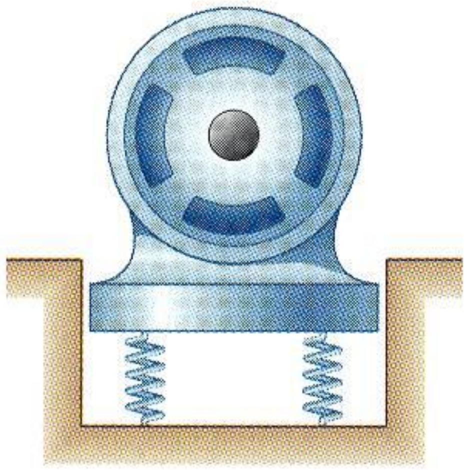
A motor of mass 200 kg is supported by four springs, each having a constant 150 kN/m. The unbalance of the motor is equivalent to a mass of 30 g located 150 mm from the axis of rotation.

Determine *a*) speed in rpm at which resonance will occur, and *b*) amplitude of the vibration at 1200 rpm.

SOLUTION:

- The resonant frequency is equal to the natural frequency of the system.
- Evaluate the magnitude of the periodic force due to the motor unbalance. Determine the vibration amplitude from the frequency ratio at 1200 rpm.

Sample Problem 19.5



m = 200 kg

k = 4(150 kg/m)

SOLUTION:

- The resonant frequency is equal to the natural frequency of the system.

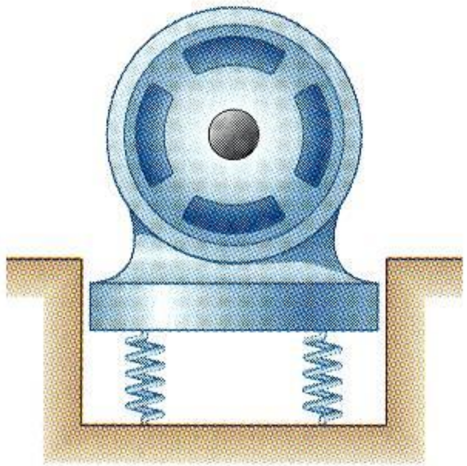
$$m = 200\ \text{kg}$$

$$k = 4(150) = 600\ \text{kN/m} = 600{,}000\ \text{N/m}$$

$$w_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{600{,}000}{200}} = 54.8\ \text{rad/s} = 523\ \text{rpm}$$

Resonance speed = 523 rpm

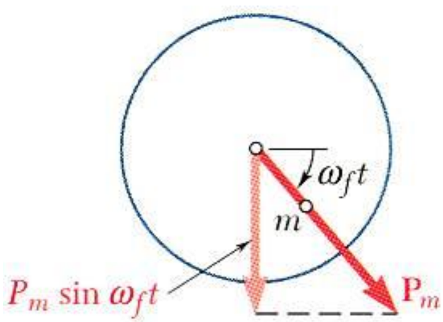
Sample Problem 19.5



$m = 200$ kg

$k = 600$ kN/m

$\omega_n = 54.8$ rad/s



- Evaluate the magnitude of the periodic force due to the motor unbalance. Determine the vibration amplitude from the frequency ratio at 1200 rpm.

$$\omega_f = \omega = 1200 \text{ rpm} = 125.7 \text{ rad/s}$$
$$m = 0.03 \text{ kg}$$

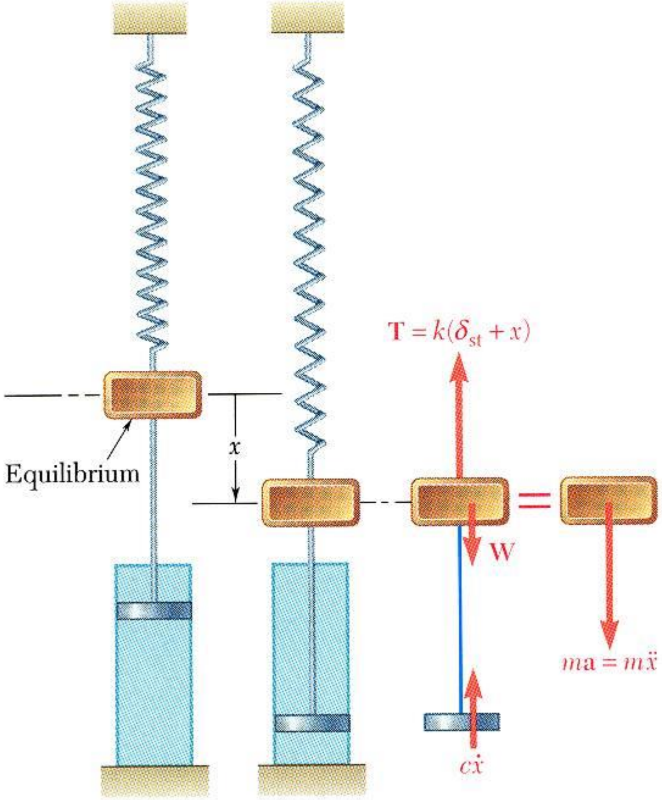
$$P_m = ma_n = mr\omega^2$$
$$= (0.03 \text{ kg})(0.15 \text{ m})(125.7)^2 = 71.1 \text{ N}$$

$$\frac{P_m}{k} = \frac{71.1 \text{ N}}{600{,}000 \text{ N/m}} \times 1000 \text{ mm} = 0.1185 \text{ mm}$$

$$x_m = \frac{P_m/k}{1-(\omega_f/\omega_n)^2} = \frac{0.1185}{1-(125.7/54.8)^2}$$
$$= -0.0278 \text{ mm}$$

$x_m = 0.0278$ mm (out of phase)

Damped Free Vibrations



- All vibrations are damped to some degree by forces due to *dry friction*, *fluid friction*, or *internal friction*.

- With *viscous damping* due to fluid friction,

$$+\downarrow \sum F = ma: \qquad W - k(\delta_{st} + x) - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

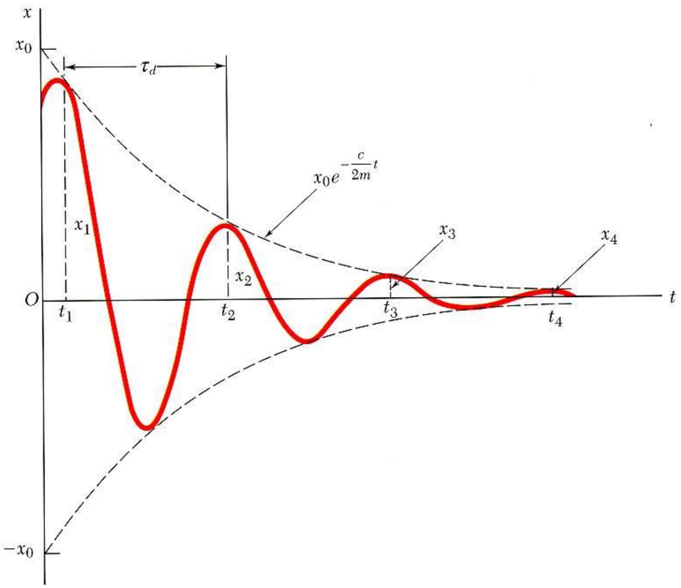
- Substituting $x = e^{\lambda t}$ and dividing through by $e^{\lambda t}$ yields the *characteristic equation*,

$$m\lambda^2 + c\lambda + k = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- Define the critical damping coefficient such that

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

Damped Free Vibrations



- Characteristic equation,

$$m\lambda^2 + c\lambda + k = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$c_c = 2m\omega_n = \text{critical damping coefficient}$$

- *Heavy damping*: $c > c_c$

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

- negative roots
- nonvibratory motion

- *Critical damping*: $c = c_c$

$$x = (C_1 + C_2 t)e^{-\omega_n t}$$

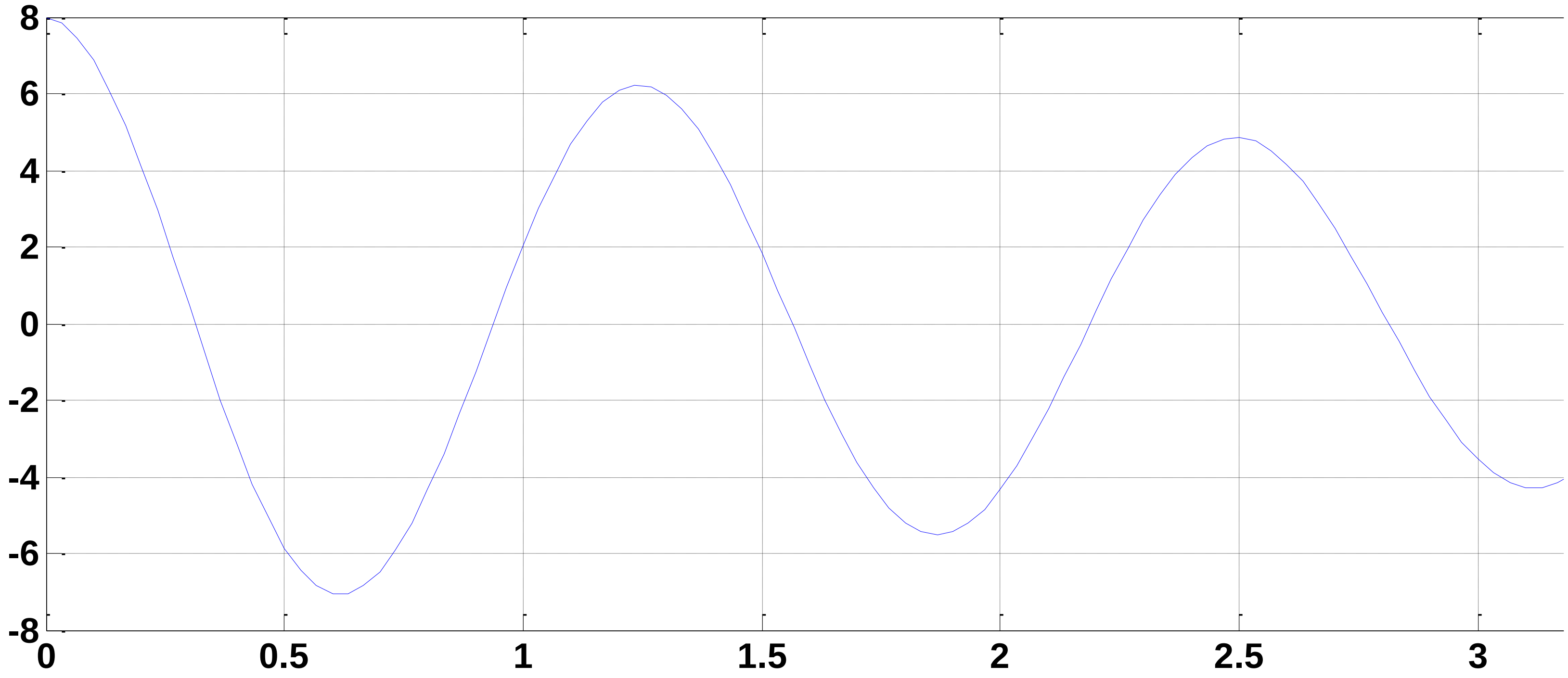
- double roots
- nonvibratory motion

- *Light damping*: $c < c_c$

$$x = e^{-(c/2m)t}\left(C_1 \sin\omega_d t + C_2 \cos\omega_d t\right)$$

$$\omega_d = \omega_n \sqrt{1 - \left(\frac{c}{c_c}\right)^2} = \text{damped frequency}$$

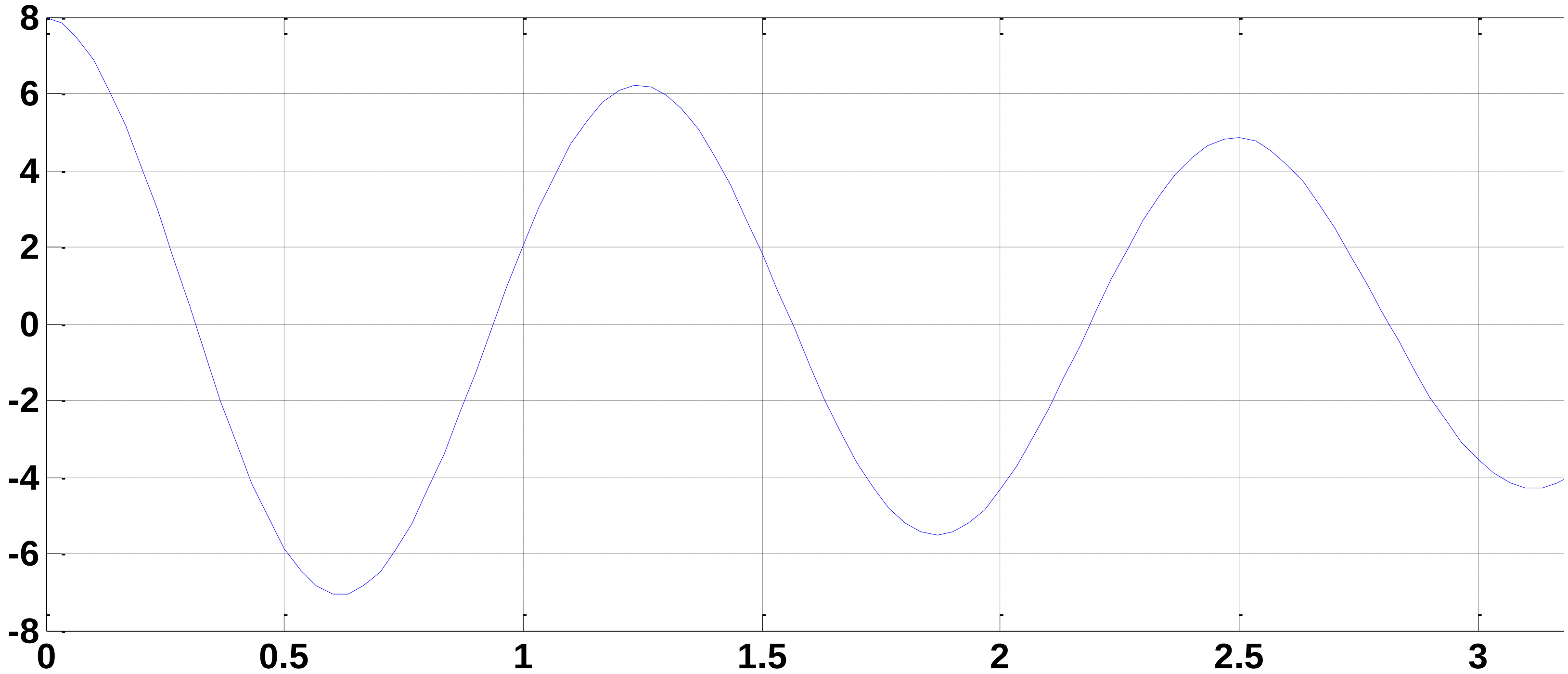
Concept Question



The graph above represents an oscillation that is…

a) Heavily damped **b) critically damped** **c) lightly damped**

Concept Question

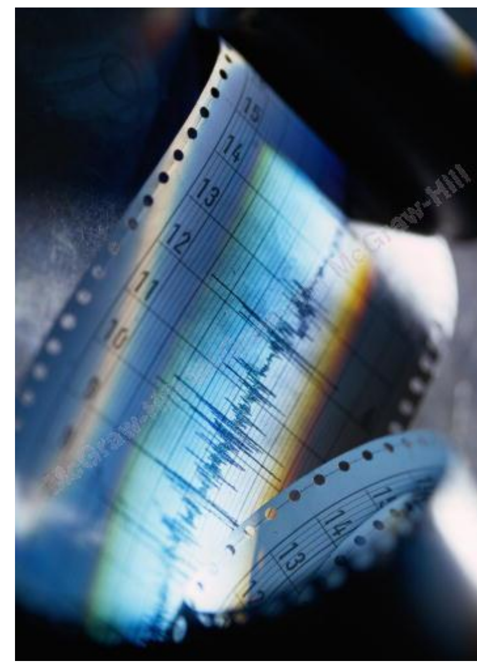


The period for the oscillation above is approximately…

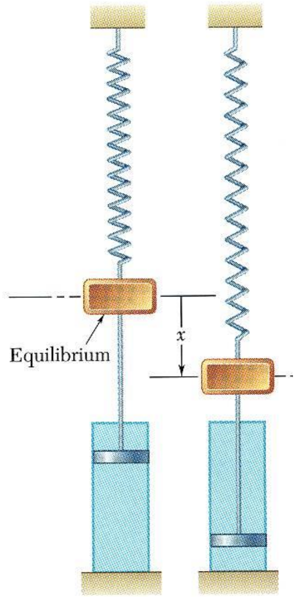
a) 1.25 seconds **b) 2.5 Hz** **c) 0.6 seconds**

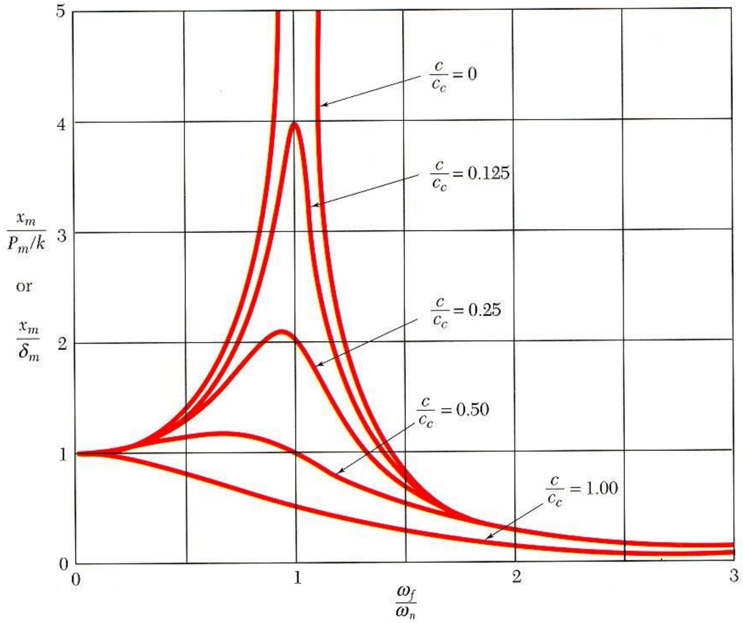
Estimate the phase shift for the oscillation **Zero**

Forced vibrations can be caused by a test machine, by rocks on a trail, by rotating machinery, and by earthquakes. Suspension systems, shock absorbers, and other energy-dissipating devices can help to dampen the resulting vibrations.



Damped Forced Vibrations



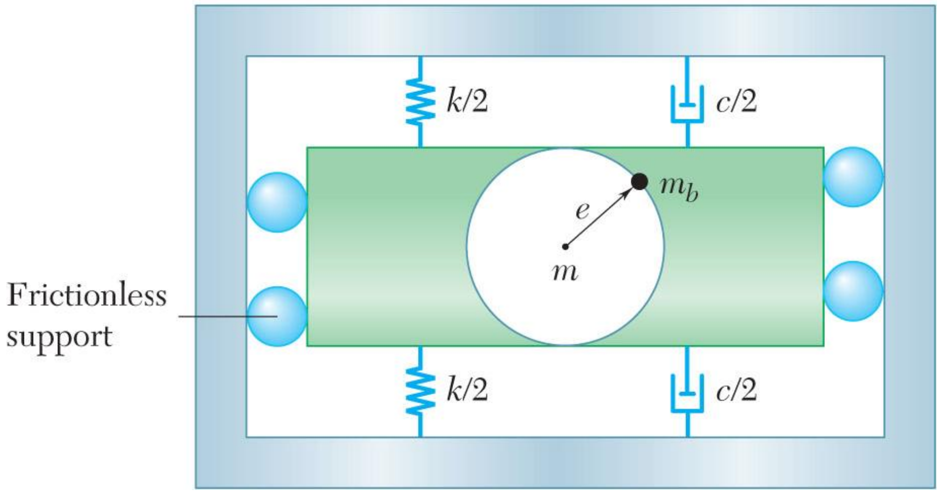


$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t \qquad x = x_{complementary} + x_{particular}$$

$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta} = \frac{1}{\sqrt{\left[1-\left(\omega_f/\omega_n\right)^2\right]^2 + \left[2\left(c/c_c\right)\left(\omega_f/\omega_n\right)\right]^2}} = \text{magnification factor}$$

$$\tan\phi = \frac{2\left(c/c_c\right)\left(\omega_f/\omega_n\right)}{1-\left(\omega_f/\omega_n\right)^2} = \text{phase difference between forcing and steady state response}$$

Group Problem Solving



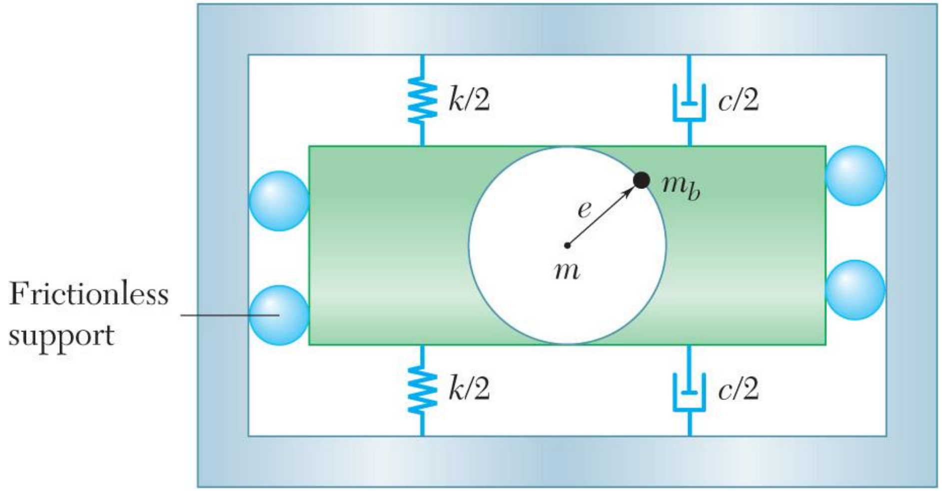
SOLUTION:

- Determine the system natural frequency, damping constant, and the unbalanced force.
- Determine the steady state response and the magnitude of the motion.

A simplified model of a washing machine is shown. A bundle of wet clothes forms a mass m_b of 10 kg in the machine and causes a rotating unbalance. The rotating mass is 20 kg (including m_b) and the radius of the washer basket e is 25 cm. Knowing the washer has an equivalent spring constant k = 1000 N/m and damping ratio $z = c/c_c = 0.05$ and during the spin cycle the drum rotates at 250 rpm, determine the amplitude of the motion.

Group Problem Solving

Given: m= 20 kg, k= 1000 N/m,
ω_f= 250 rpm, e= 25 cm, m_b= 10 kg
Find: x_m



Calculate the forced circular frequency and the natural circular frequency

$$\omega_f = \frac{(2\pi)(250)}{60} = 26.18 \text{ rad/s}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{20}} = 7.0711 \text{ rad/s}$$

Calculate the critical damping constant c_c and the damping constant c

$$c_c = 2\sqrt{km} = 2\sqrt{(1000)(20)} = 282.84 \text{ N}\cdot\text{s/m}$$

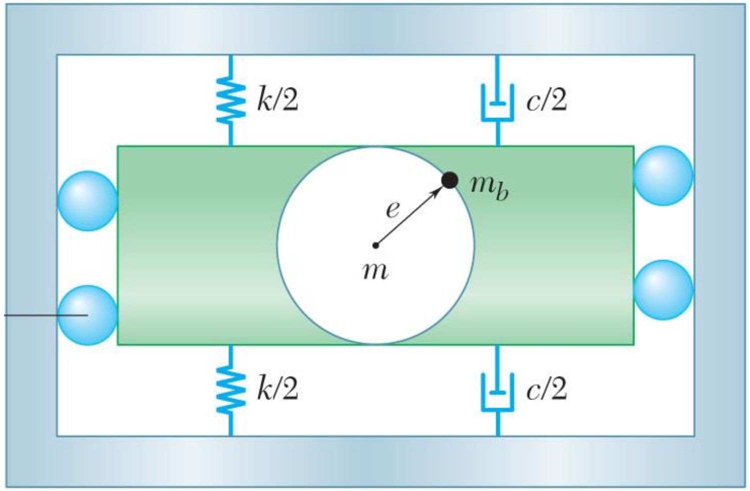
$$c = \left(\frac{c}{c_c}\right)c = (0.05)(141.42) = 14.1421 \text{ N}\cdot\text{s/m}$$

Group Problem Solving

Calculate the unbalanced force caused by the wet clothes

$$P_m = m_b e \omega_f^2$$

$$P_m = (10\text{ kg})(0.25\text{ m})(26.18\text{ rad/s})^2 = 1713.48\text{ N}$$



Use Eq 19.52 to determine x_m

$$m\ddot{x} + c\dot{x} + kx = P_m \sin w_f t$$

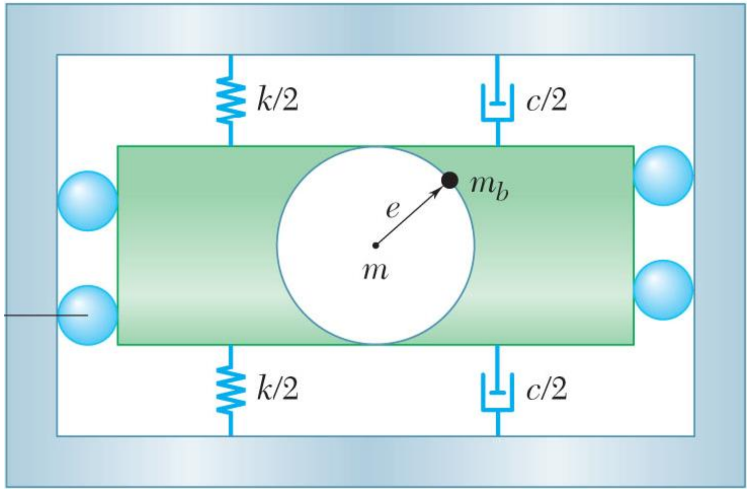
$$x_m = \frac{P_m}{\sqrt{\left(k - m\omega_f^2\right)^2 + (c\omega_f)^2}}$$

$$= \frac{1713.48}{\sqrt{[1000 - (20)(26.18)^2]^2 + [(141421)(26.18)]^2}}$$

$$= \frac{1713.48}{\sqrt{(-12,707.8)^2 + (370.24)^2}} = \frac{1713.48}{12,713.2} = 0.13478\text{ m}$$

$$x_m = 134.8\text{ mm}$$

Concept Question



The following parameters were found in the previous problem:

$\omega_f = 26.18 \text{ rad/s}$

$\omega_n = 7.0711 \text{ rad/s}$

$\zeta = 0.05$

What would happen to the amplitude x_m if the forcing frequency ω_f was cut in half?

a) **The vibration amplitude remains the same.**
b) **The vibration amplitude would increase.**
c) **The vibration amplitude would decrease.**

Concept Question

Case 1

$$\omega_f = 26.18 \text{ rad/s}$$

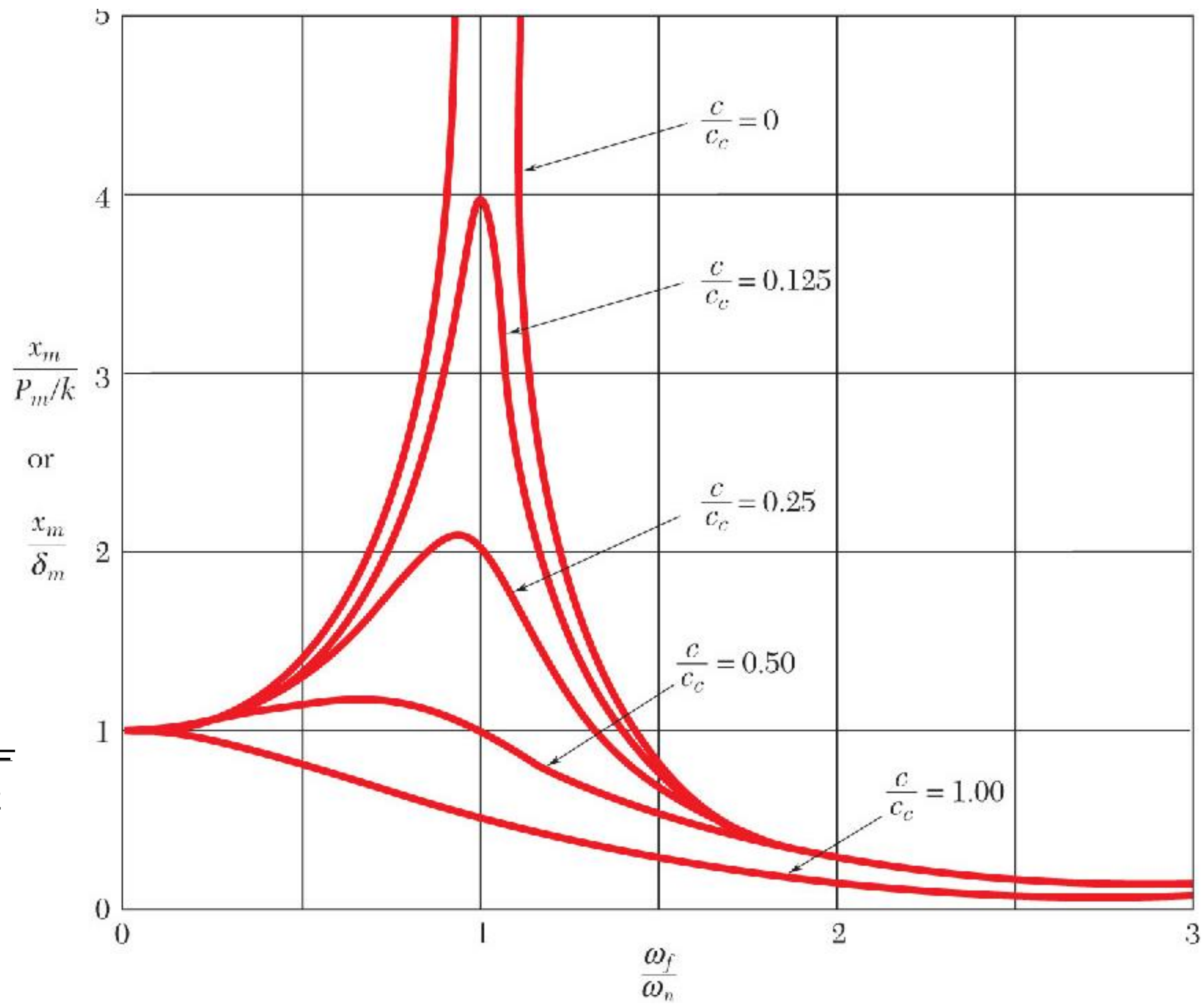
$$\omega_n = 7.0711 \text{ rad/s}$$

Case 2

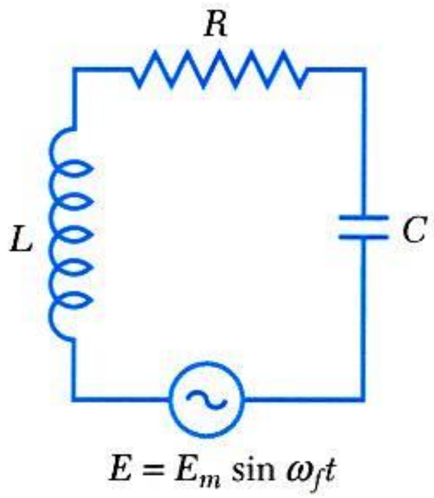
$$\omega_f = 13.09 \text{ rad/s}$$

$$\omega_n = 7.0711 \text{ rad/s}$$

$$x_m = \frac{P_m}{\sqrt{\left(k - m\omega_f^2\right)^2 + (c\omega_f)^2}}$$



Electrical Analogues



- Consider an electrical circuit consisting of an inductor, resistor and capacitor with a source of alternating voltage

$$E_m \sin \omega_f t - L\frac{di}{dt} - Ri - \frac{q}{C} = 0$$

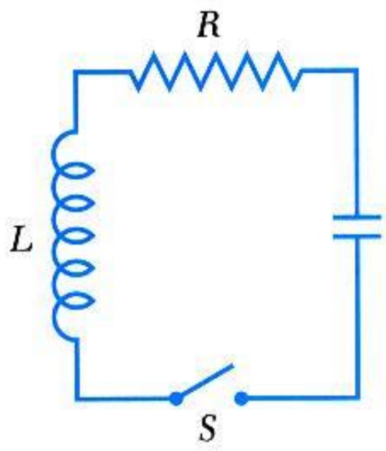
$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E_m \sin \omega_f t$$

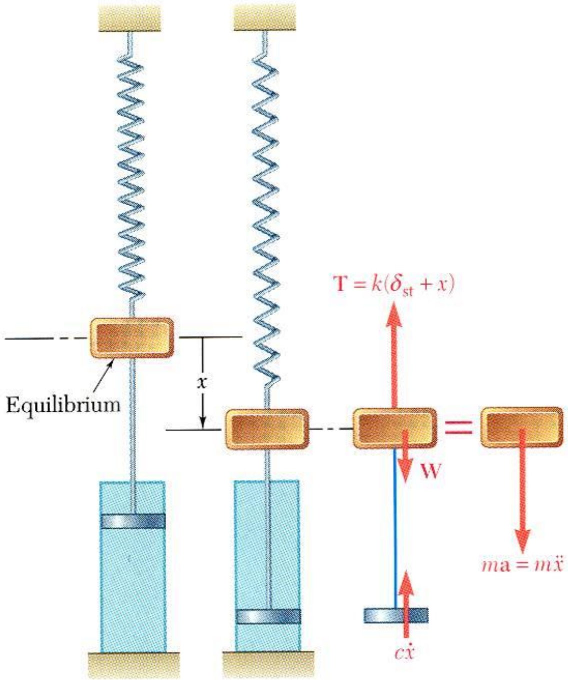
- Oscillations of the electrical system are analogous to damped forced vibrations of a mechanical system.

Table 19.2. Characteristics of a Mechanical System and of Its Electrical Analogue

Mechanical System		Electrical Circuit	
m	Mass	L	Inductance
c	Coefficient of viscous damping	R	Resistance
k	Spring constant	$1/C$	Reciprocal of capacitance
x	Displacement	q	Charge
v	Velocity	i	Current
P	Applied force	E	Applied voltage

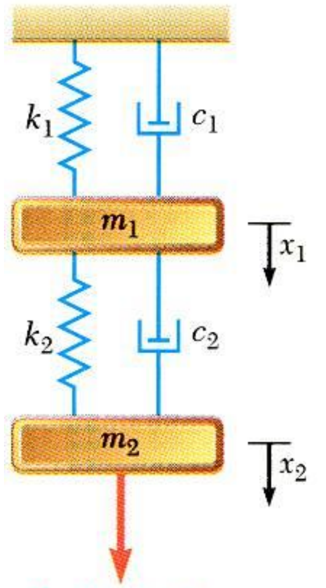
Electrical Analogues

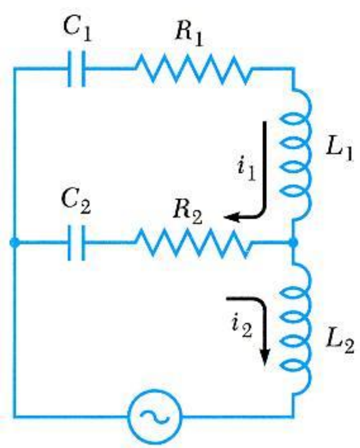




- The analogy between electrical and mechanical systems also applies to transient as well as steady-state oscillations.
- With a charge $q = q_0$ on the capacitor, closing the switch is analogous to releasing the mass of the mechanical system with no initial velocity at $x = x_0$.
- If the circuit includes a battery with constant voltage E, closing the switch is analogous to suddenly applying a force of constant magnitude P to the mass of the mechanical system.

Electrical Analogues





- The electrical system analogy provides a means of experimentally determining the characteristics of a given mechanical system.

- For the mechanical system,

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1 x_1 + k_2(x_1 - x_2) = 0$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = P_m \sin \omega_f t$$

- For the electrical system,

$$L_1\ddot{q}_1 + R_1(\dot{q}_1 - \dot{q}_2) + \frac{q_1}{C_1} + \frac{q_1 - q_2}{C_2} = 0$$

$$L_2\ddot{q}_2 + R_2(\dot{q}_2 - \dot{q}_1) + \frac{q_2 - q_1}{C_2} = E_m \sin \omega_f t$$

- The governing equations are equivalent. The characteristics of the vibrations of the mechanical system may be inferred from the oscillations of the electrical system.