Tenth Edition

# VECTOR MECHANICS FOR ENGINEERS:

CHAPTER

15

# DYNAMICS

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## Kinematics of Rigid Bodies

# Contents

## Applications

**A battering ram is an example of curvilinear translation – the ram stays horizontal as it swings through its motion.**

# Applications

**How can we determine the velocity of the tip of a turbine blade?**

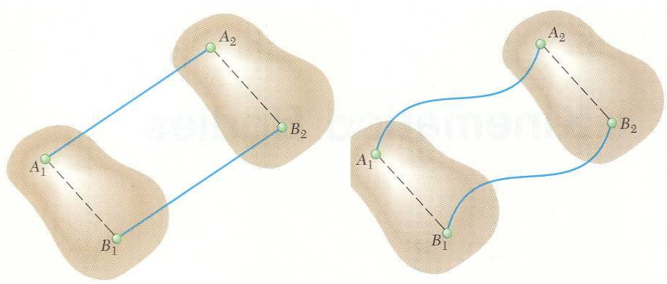
## Applications

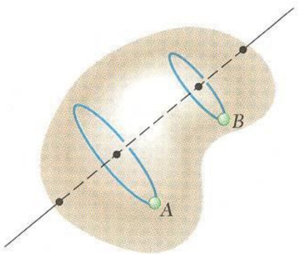
**Planetary gear systems are used to get high reduction ratios with minimum weight and space. How can we design the correct gear ratios?**

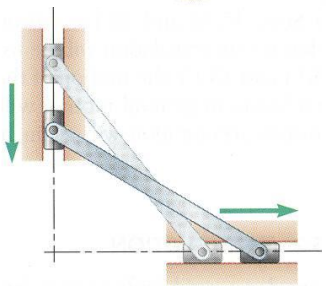
## Applications

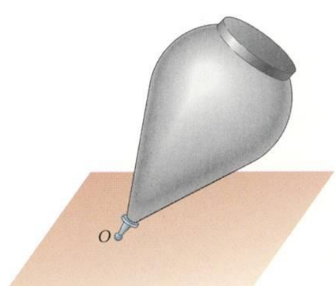
**Biomedical engineers must determine the velocities and accelerations of the leg in order to design prostheses.**

# Introduction



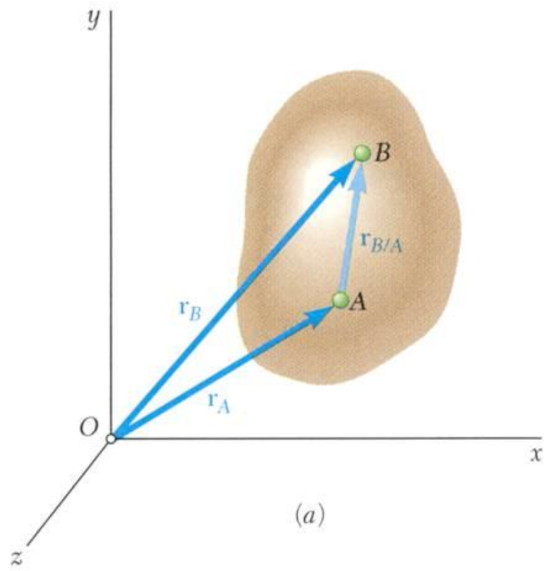



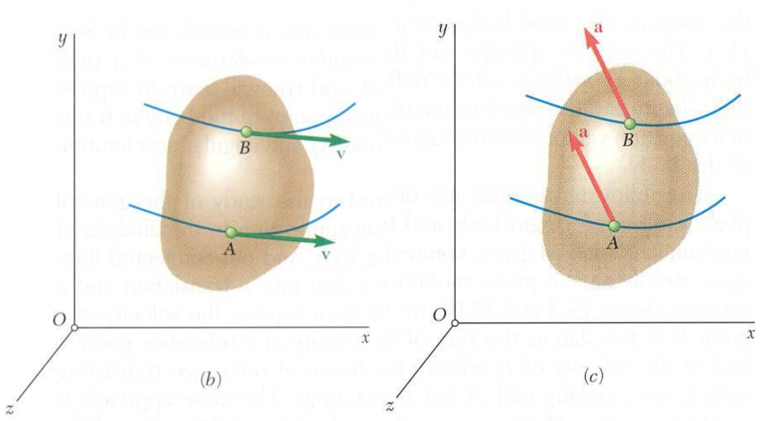






- Kinematics of rigid bodies: relations between time and the positions, velocities, and accelerations of the particles forming a rigid body.
- Classification of rigid body motions:
  - translation:
    - rectilinear translation
    - curvilinear translation
  - rotation about a fixed axis
  - general plane motion
  - motion about a fixed point
  - general motion

## Translation




- Consider rigid body in translation:
  - direction of any straight line inside the body is constant,
  - all particles forming the body move in parallel lines.
- For any two particles in the body,

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \quad (\vec{r}_{B/A} \text{ being constant})$$

- Differentiating with respect to time,

$$\dot{\vec{r}}_B = \dot{\vec{r}}_A + \dot{\vec{r}}_{B/A} = \dot{\vec{r}}_A$$

$$\vec{v}_B = \vec{v}_A$$

All particles have the same velocity.

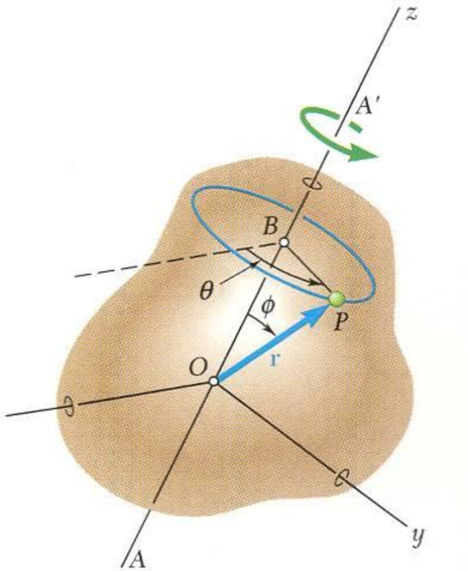
- Differentiating with respect to time again,

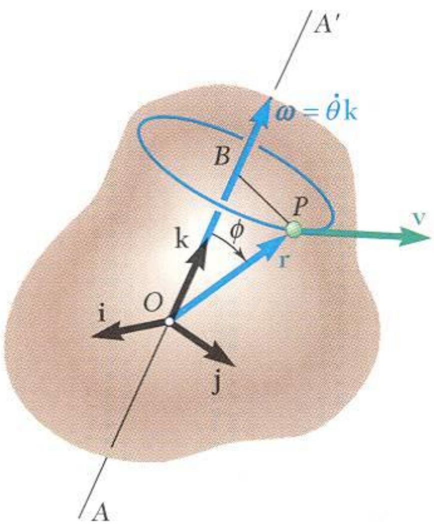
$$\ddot{\vec{r}}_B = \ddot{\vec{r}}_A + \ddot{\vec{r}}_{B/A} = \ddot{\vec{r}}_A$$

$$\vec{a}_B = \vec{a}_A$$

All particles have the same acceleration.

## Rotation About a Fixed Axis. Velocity




- Consider rotation of rigid body about a fixed axis *AA'*

- Velocity vector $\vec{v} = d\vec{r}/dt$ of the particle *P* is tangent to the path with magnitude $v = ds/dt$

$$\Delta s = (BP)\Delta\theta = (r\sin\phi)\Delta\theta$$

$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0}(r\sin\phi)\frac{\Delta\theta}{\Delta t} = r\dot{\theta}\sin\phi$$

- The same result is obtained from

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$$

$$\vec{\omega} = \omega\vec{k} = \dot{\theta}\vec{k} = \textit{angular velocity}$$

# Concept Quiz

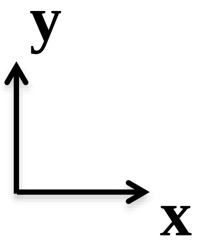
**What is the direction of the velocity of point A on the turbine blade?**

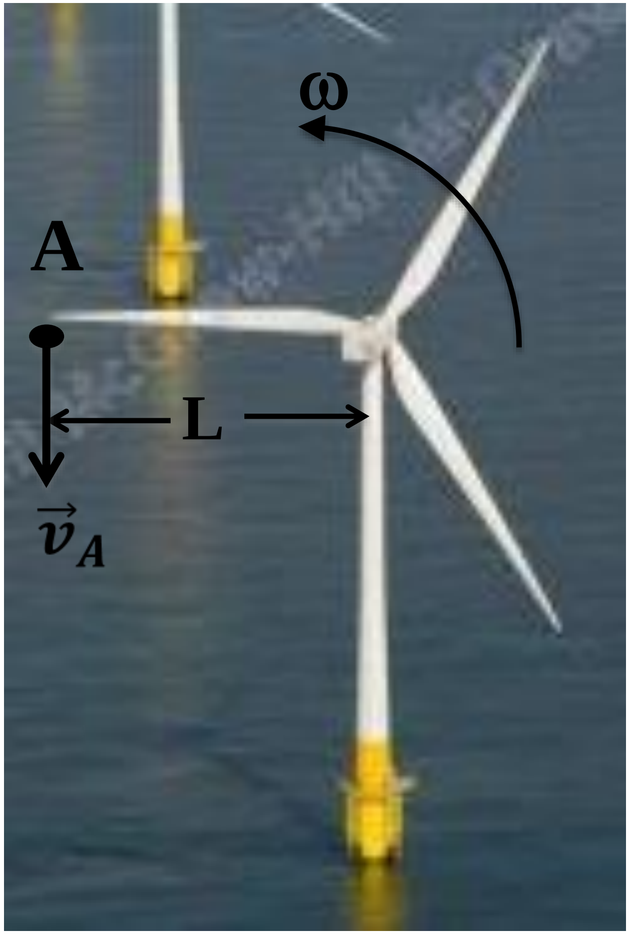
a) →

b) ←

c) ↑

d) ↓



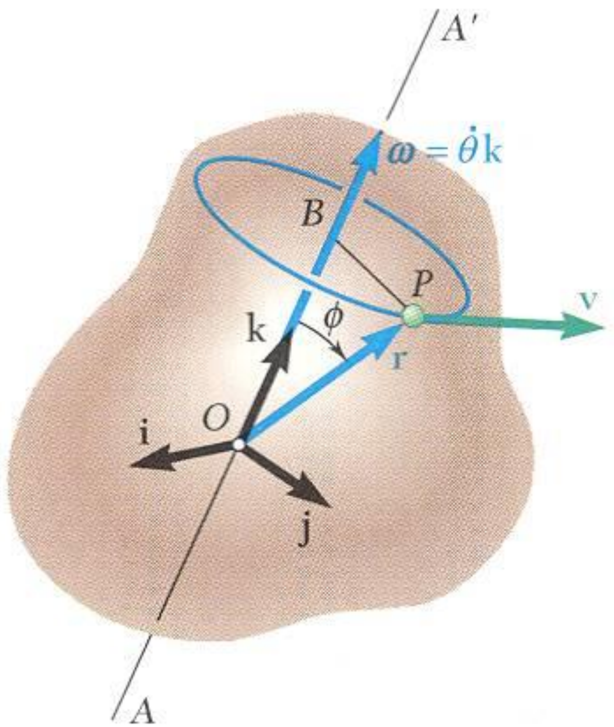




$$\vec{v}_A = \vec{\omega} \times \vec{r}$$

$$\vec{v}_A = \omega\hat{k} \times -L\hat{i}$$

$$\vec{v}_A = -L\omega\hat{j}$$

# Rotation About a Fixed Axis. Acceleration




- Differentiating to determine the acceleration,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$
$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$
$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}$$

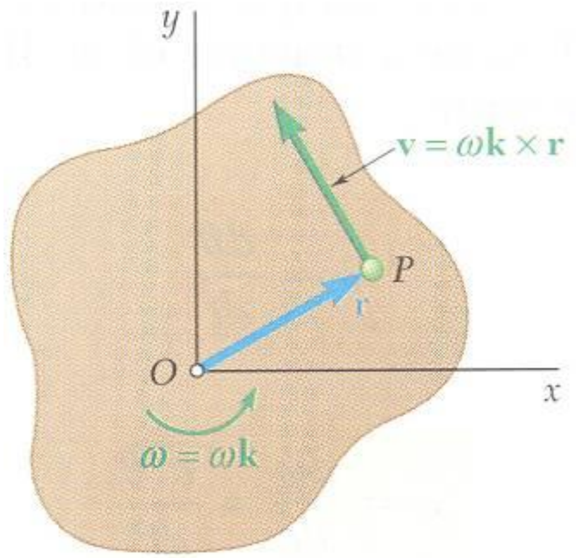
- $\frac{d\vec{\omega}}{dt} = \vec{\alpha} = angular\ acceleration$

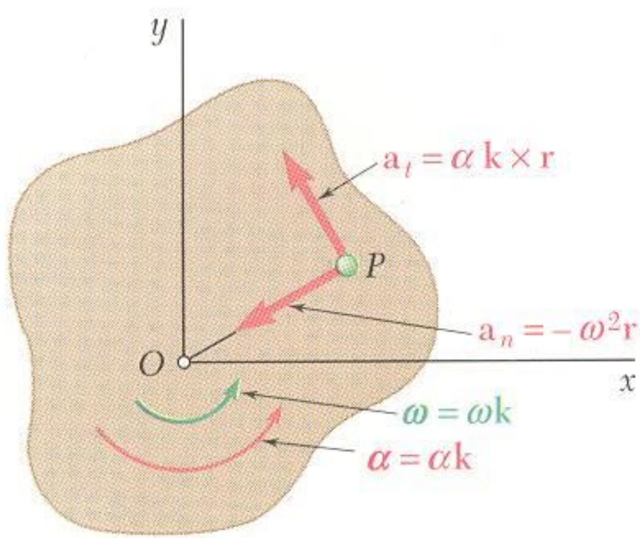
$$= \alpha\vec{k} = \dot{\omega}\vec{k} = \ddot{\theta}\vec{k}$$

- Acceleration of $P$ is combination of two vectors,

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\omega} \times \vec{r}$$
$$\vec{\alpha} \times \vec{r} = \text{tangential acceleration component}$$
$$\vec{\omega} \times \vec{\omega} \times \vec{r} = \text{radial acceleration component}$$

# Rotation About a Fixed Axis. Representative Slab




- Consider the motion of a representative slab in a plane perpendicular to the axis of rotation.

- Velocity of any point $P$ of the slab,

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega \vec{k} \times \vec{r}$$
$$v = r\omega$$

- Acceleration of any point $P$ of the slab,

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\omega} \times \vec{r}$$
$$= \alpha \vec{k} \times \vec{r} - \omega^2 \vec{r}$$

- Resolving the acceleration into tangential and normal components,

$$\vec{a}_t = \alpha \vec{k} \times \vec{r} \qquad a_t = r\alpha$$
$$\vec{a}_n = -\omega^2 \vec{r} \qquad a_n = r\omega^2$$

## Concept Quiz

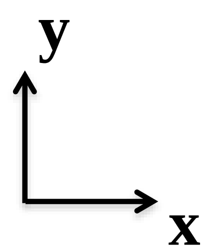
**What is the direction of the normal acceleration of point A on the turbine blade?**

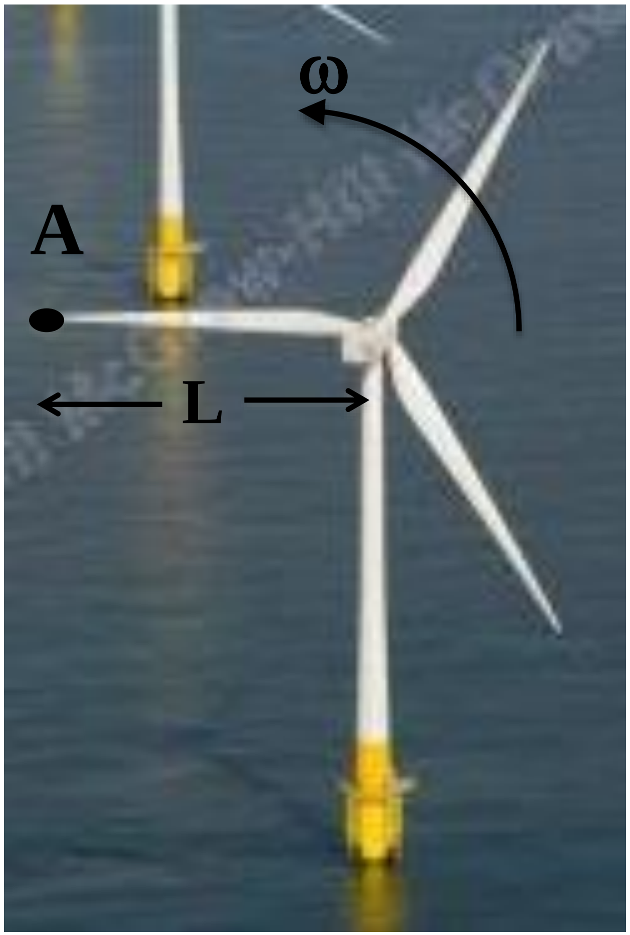
a) → (circled)

b) ←

c) ↑

d) ↓



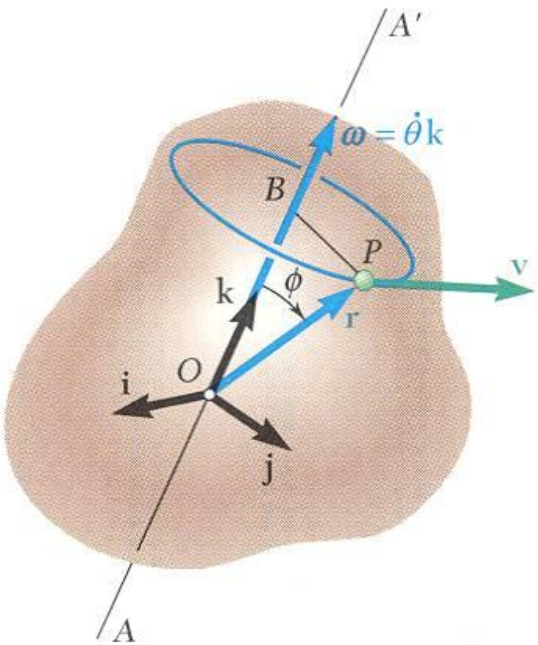




$$\vec{a}_n = -\omega^2 \vec{r}$$

$$\vec{a}_n = -\omega^2 (-L\hat{i})$$

$$\vec{a}_n = L\omega^2 \hat{i}$$

## Equations Defining the Rotation of a Rigid Body About a Fixed Axis




- Motion of a rigid body rotating around a fixed axis is often specified by the type of angular acceleration.

- Recall $\omega = \dfrac{d\theta}{dt}$ or $dt = \dfrac{d\theta}{\omega}$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}$$

- *Uniform Rotation*, $\alpha = 0$:

$$\theta = \theta_0 + \omega t$$

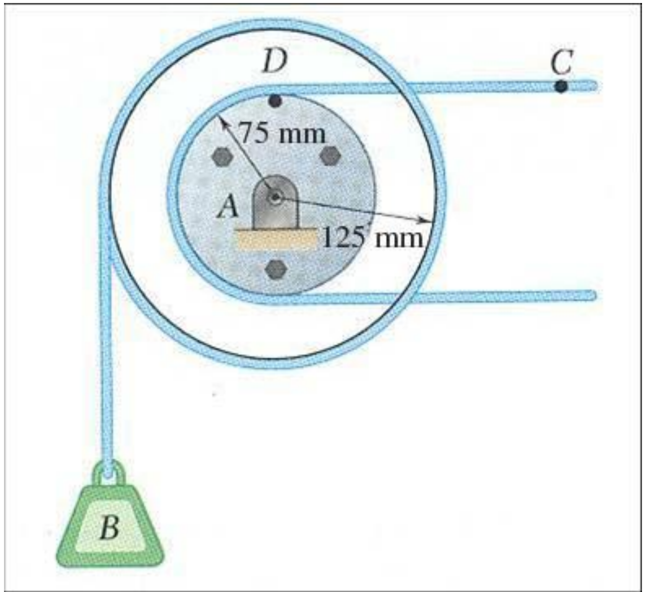
- *Uniformly Accelerated Rotation*, $\alpha$ = constant:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

## Sample Problem 5.1



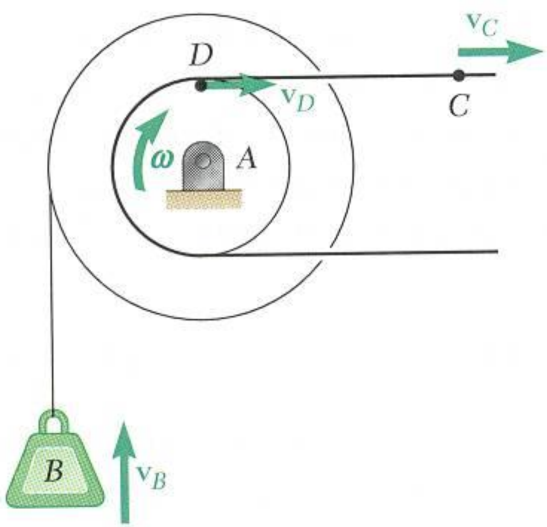

Cable *C* has a constant acceleration of 225 m/s$^2$ and an initial velocity of 300 mm/s, both directed to the right.

Determine *(a)* the number of revolutions of the pulley in 2 s, *(b)* the velocity and change in position of the load *B* after 2 s, and *(c)* the acceleration of the point *D* on the rim of the inner pulley at $t$ = 0.

SOLUTION:

- Due to the action of the cable, the tangential velocity and acceleration of *D* are equal to the velocity and acceleration of *C*. Calculate the initial angular velocity and acceleration.
- Apply the relations for uniformly accelerated rotation to determine the velocity and angular position of the pulley after 2 s.
- Evaluate the initial tangential and normal acceleration components of *D*.

# Sample Problem 5.1




<u>SOLUTION</u>:

- The tangential velocity and acceleration of *D* are equal to the velocity and acceleration of *C*.

$$(\vec{v}_D)_0 = (\vec{v}_C)_0 = 300\,\text{mm/s} \rightarrow \qquad (\vec{a}_D)_t = \vec{a}_C = 225\,\text{mm/s}^2 \rightarrow$$

$$(v_D)_0 = r\omega_0 \qquad (a_D)_t = r\alpha$$

$$\omega_0 = \frac{(v_D)_0}{r} = \frac{300}{75} = 4\,\text{rad/s}\,\boldsymbol{i} \qquad \alpha = \frac{(a_D)_t}{r} = \frac{225}{75} = 3\,\text{rad/s}^2\,\boldsymbol{i}$$

- Apply the relations for uniformly accelerated rotation to determine velocity and angular position of pulley after 2 s.

$$\omega = \omega_0 + \alpha t = 4\,\text{rad/s} + (3\,\text{rad/s}^2)(2\,\text{s}) = 10\,\text{rad/s}$$

$$q = w_0 t + \tfrac{1}{2} a t^2 = (4\,\text{rad/s})(2\ \text{s}) + \tfrac{1}{2}(3\,\text{rad/s}^2)(2\ \text{s})^2$$

$$= 14\ \text{radi}$$

$$N = (14\,\text{rad})\left(\frac{1\,\text{rev}}{2\pi\,\text{rad}}\right) = \text{number of revs}$$

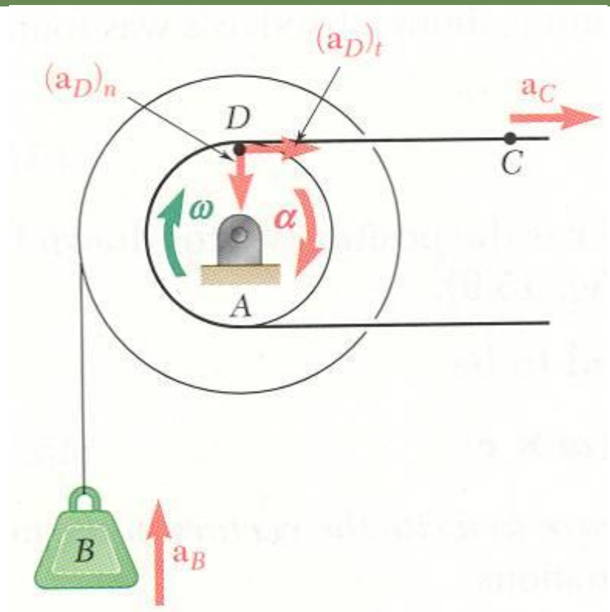
$$\boxed{N = 2.23\,\text{rev}}$$

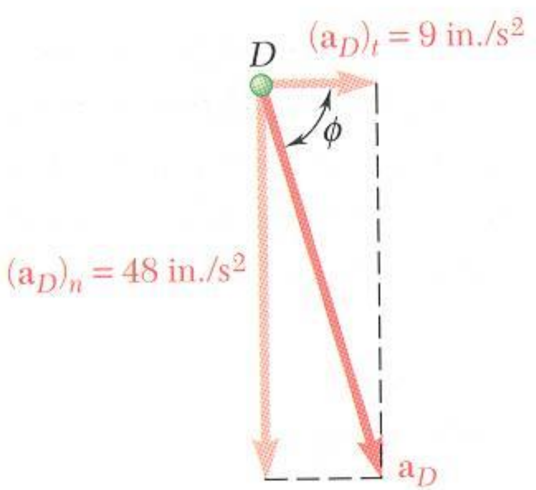
$$v_B = rw = (125\ \text{mm})(10\,\text{rad/s})$$

$$Dy_B = rq = (125\ \text{mm})(14\ \text{rad})$$

$$\boxed{\vec{v}_B = 1.25\,\text{m/s} - \quad Dy_B = 1.75\ \text{m} -}$$

# Sample Problem 5.1




- Evaluate the initial tangential and normal acceleration components of *D*.

$$(\vec{a}_D)_t = \vec{a}_C = 25\,\text{mm/s}^2 \rightarrow$$

$$(a_D)_n = r_D w_0^2 = (75\,\text{mm})(4\,\text{rad/s})^2 = 1200\,\text{mm/s}^2$$

$$(\vec{a}_D)_t = 225\,\text{mm/s}^2 \rightarrow \qquad (\vec{a}_D)_n = 1200\,\text{mm/s}^2 \downarrow$$

Magnitude and direction of the total acceleration,

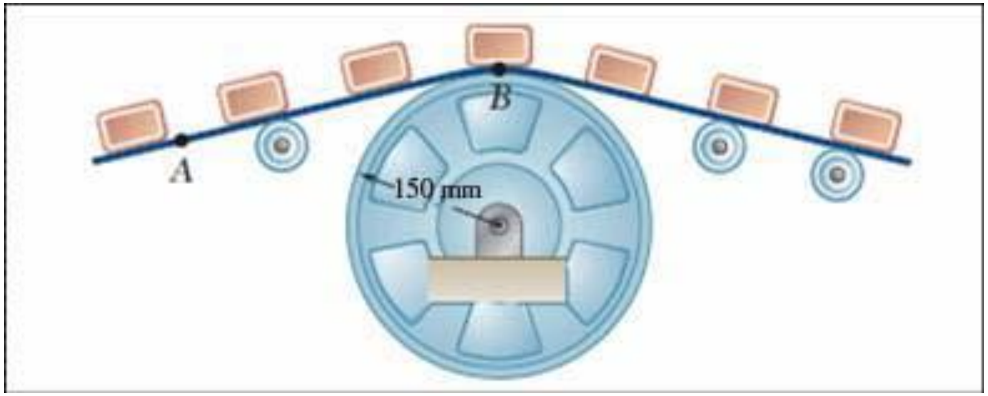
$$a_D = \sqrt{(a_D)_t^2 + (a_D)_n^2}$$
$$= \sqrt{(225)^2 + (1200)^2}$$
$$= 1220.9\,\text{mm/s}^2$$

$$a_D = 1.221\,\text{m/s}^2$$

$$\tan f = \frac{(a_D)_n}{(a_D)_t}$$
$$= \frac{1200}{225}$$

$$\phi = 79.4°$$

# Group Problem Solving




A series of small machine components being moved by a conveyor belt pass over a 150 mm-radius idler pulley. At the instant shown, the velocity of point *A* is 375 mm/s to the left and its acceleration is 225 mm/s$^2$ to the right. Determine (*a*) the angular velocity and angular acceleration of the idler pulley, (*b*) the total acceleration of the machine
component at *B*.

SOLUTION:

- Using the linear velocity and accelerations, calculate the angular velocity and acceleration.
- Using the angular velocity, determine the normal acceleration.
- Determine the total acceleration using the tangential and normal acceleration components of *B*.

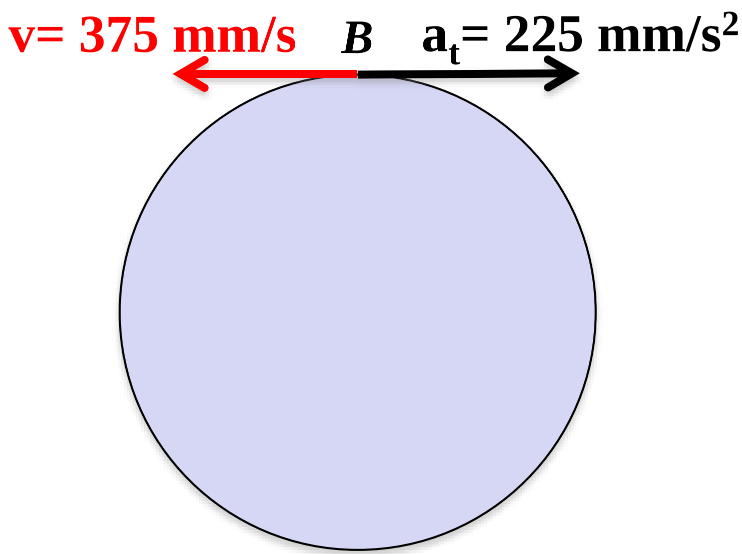
## Group Problem Solving

**Find the angular velocity of the idler pulley using the linear velocity at B.**

$$v = r\omega$$

$$375 \text{ mm/s} = (150 \text{ mm})\omega$$

$$\boldsymbol{\omega} = 2.50 \text{ rad/s} \circlearrowright$$




**Find the angular acceleration of the idler pulley using the linear velocity at B.**

$$a = r\alpha$$

$$225 \text{ mm/s}^2 = (150 \text{ mm})\alpha$$

$$\boldsymbol{\alpha} = 1.500 \text{ rad/s}^2 \circlearrowright$$

**Find the normal acceleration of point *B*.**

$$a_n = r\omega^2$$

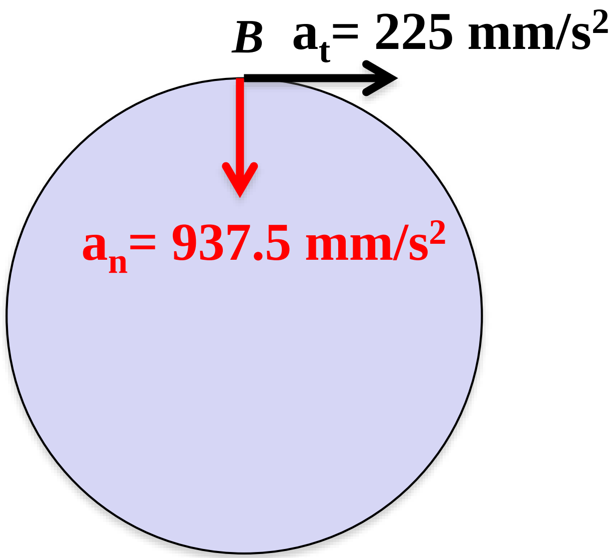
$$= (150 \text{ mm})(2.5 \text{ rad/s})^2$$

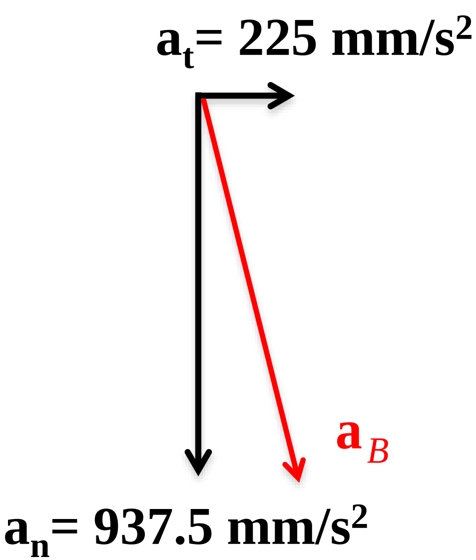
$$\mathbf{a}_n = 937.5 \text{ mm/s}^2$$

**What is the direction of the normal acceleration of point *B*?**

Downwards, towards the center

# Group Problem Solving




**Find the total acceleration of the machine component at point *B*.**

$$\mathbf{a}_t = 225\ \text{mm/s}^2 \qquad \mathbf{a}_n = 937.5\ \text{mm/s}^2$$

**Calculate the magnitude**

$$|\mathbf{a}| = \sqrt{(225)^2 + (937.5)^2} = 964\ \text{mm/s}^2$$

**Calculate the angle from the horizontal**

$$\theta = \arctan\left(\frac{937.5}{225}\right) = 76.5°$$

**Combine for a final answer**

$$\mathbf{a}_B = 964\ \text{mm/s}^2 \quad 76.5°$$

# Golf Robot

Not ours – maybe Tom Mase has pic?


**A golf robot is used to test new equipment. If the angular velocity of the arm is doubled, what happens to the normal acceleration of the club head?**

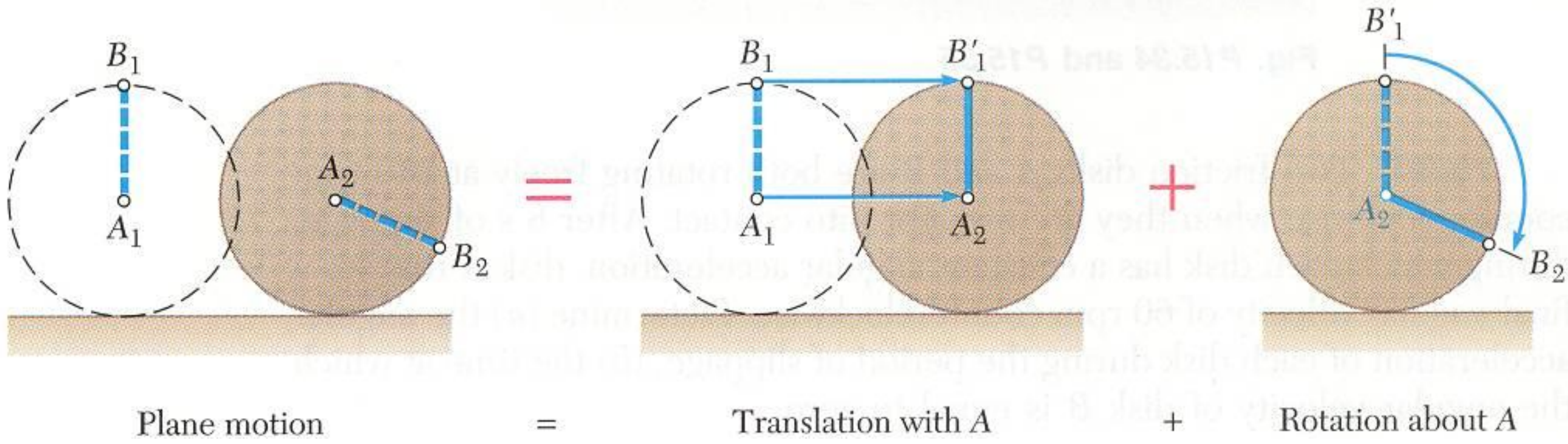
**If the arm is shortened to ¾ of its original length, what happens to the tangential acceleration of the club head?**

**If the speed of the club head is constant, does the club head have any linear accelertion ?**

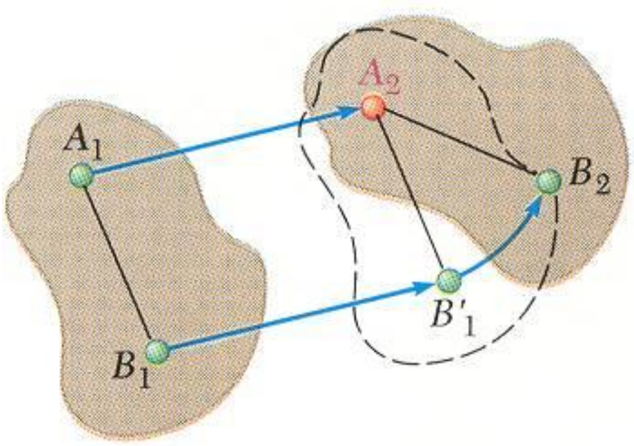
## Example – General Plane Motion

**The knee has linear velocity and acceleration from both translation (the runner moving forward) as well as rotation (the leg rotating about the hip).**

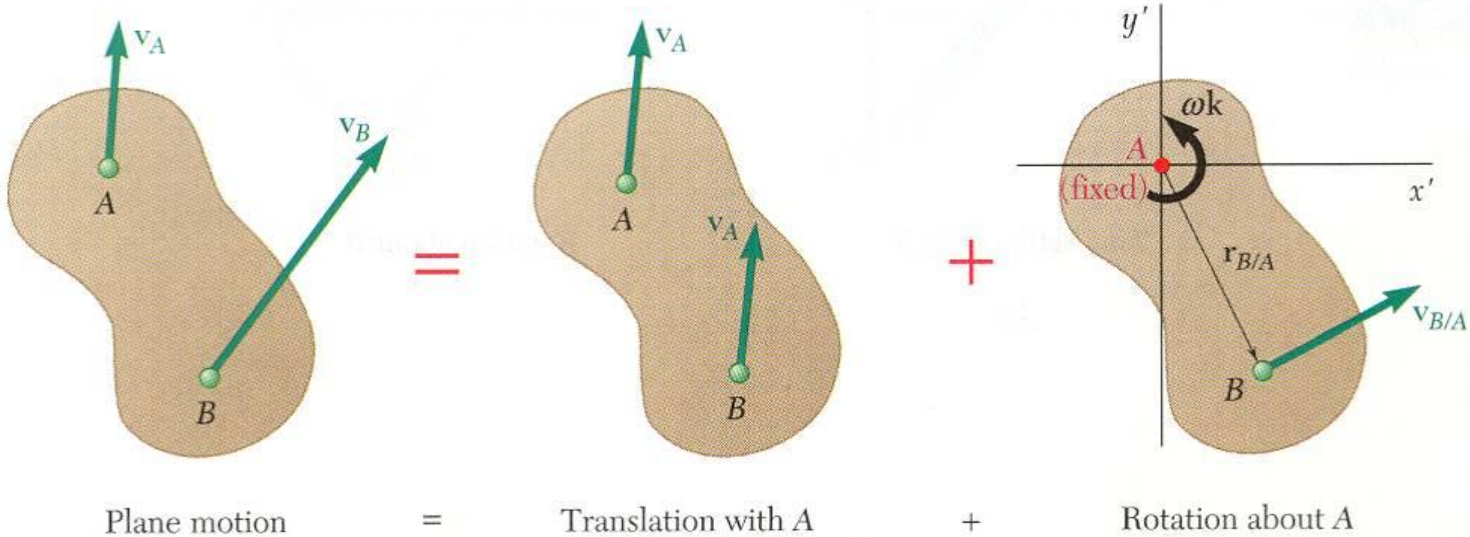
# General Plane Motion

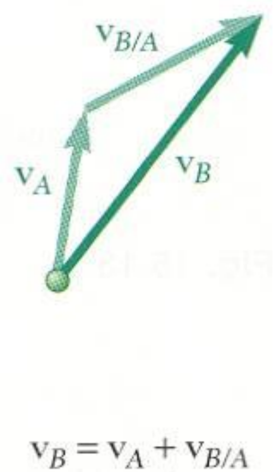



- *General plane motion* is neither a translation nor a rotation.
- General plane motion can be considered as the *sum* of a translation and rotation.
- Displacement of particles *A* and *B* to $A_2$ and $B_2$ can be divided into two parts:
  - translation to $A_2$ and $B_1'$
  - rotation of $B_1'$ about $A_2$ to $B_2$



# Absolute and Relative Velocity in Plane Motion



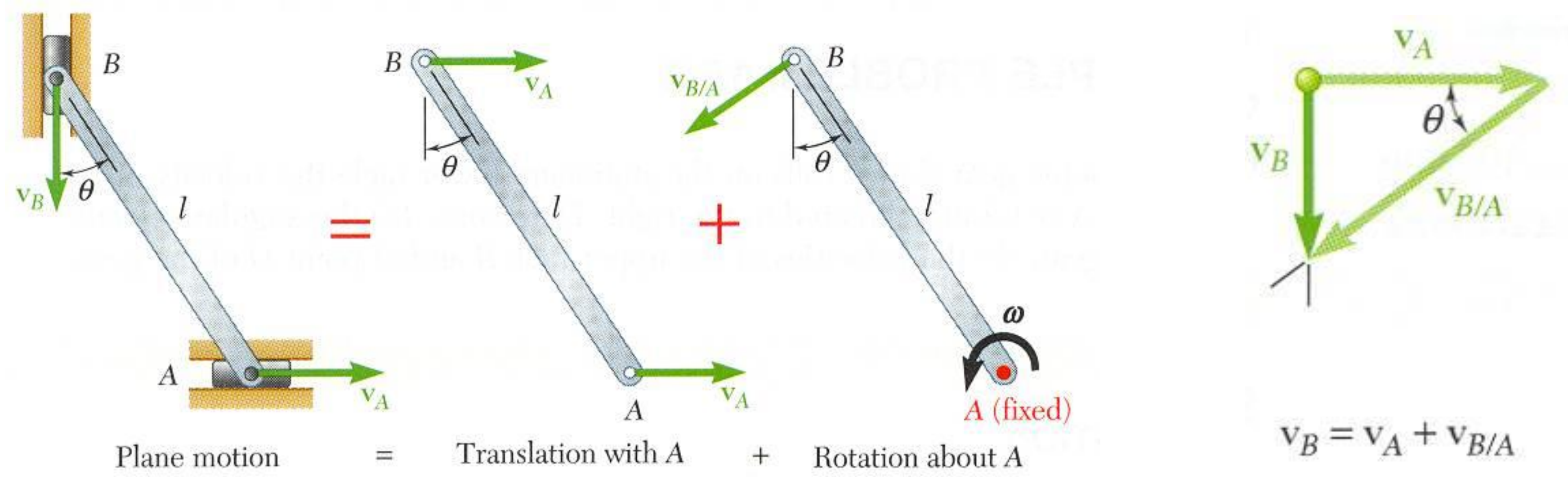

- Any plane motion can be replaced by a translation of an arbitrary reference point *A* and a simultaneous rotation about *A*.

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

$$\vec{v}_{B/A} = \omega\vec{k} \times \vec{r}_{B/A} \qquad v_{B/A} = r\omega$$

$$\vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$

# Absolute and Relative Velocity in Plane Motion




- Assuming that the velocity $v_A$ of end $A$ is known, wish to determine the velocity $v_B$ of end $B$ and the angular velocity ω in terms of $v_A$, $l$, and θ.
- The direction of $v_B$ and $v_{B/A}$ are known. Complete the velocity diagram.

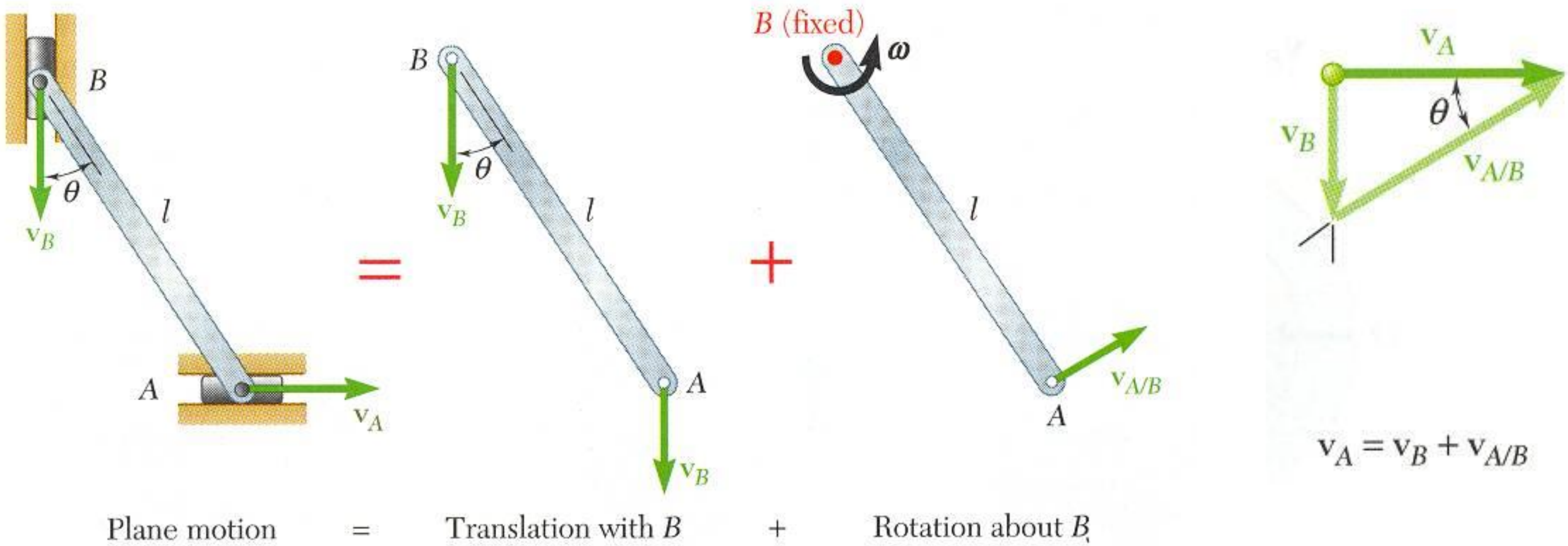
$$\frac{v_B}{v_A} = \tan\theta$$

$$v_B = v_A \tan\theta$$

$$\frac{v_A}{v_{B/A}} = \frac{v_A}{l\omega} = \cos\theta$$

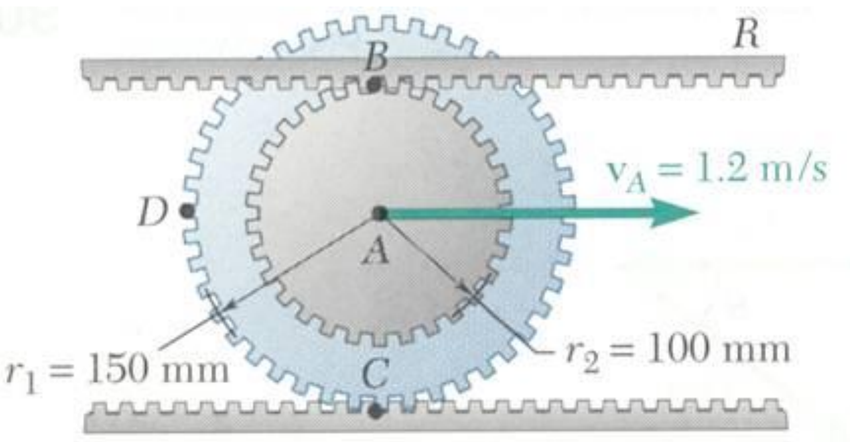
$$\omega = \frac{v_A}{l\cos\theta}$$

# Absolute and Relative Velocity in Plane Motion




- Selecting point *B* as the reference point and solving for the velocity $v_A$ of end *A* and the angular velocity $\omega$ leads to an equivalent velocity triangle.
- $v_{A/B}$ has the same magnitude but opposite sense of $v_{B/A}$. The sense of the relative velocity is dependent on the choice of reference point.
- Angular velocity $\omega$ of the rod in its rotation about *B* is the same as its rotation about *A*. Angular velocity is not dependent on the choice of reference point.

## Sample Problem 15.2




The double gear rolls on the stationary lower rack: the velocity of its center is 1.2 m/s.

Determine *(a)* the angular velocity of the gear, and *(b)* the velocities of the upper rack *R* and point *D* of the gear.

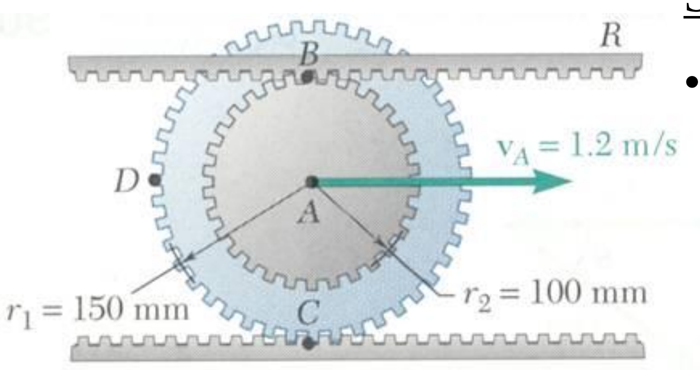
SOLUTION:

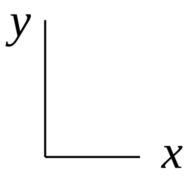
- The displacement of the gear center in one revolution is equal to the outer circumference. Relate the translational and angular displacements. Differentiate to relate the translational and angular velocities.
- The velocity for any point *P* on the gear may be written as

$$\vec{v}_P = \vec{v}_A + \vec{v}_{P/A} = \vec{v}_A + \omega\vec{k} \times \vec{r}_{P/A}$$

Evaluate the velocities of points *B* and *D*.

## Sample Problem 15.2




SOLUTION:

- The displacement of the gear center in one revolution is equal to the outer circumference.

For $x_A > 0$ (moves to right), $\omega < 0$ (rotates clockwise).

$$\frac{x_A}{2\pi r} = -\frac{\theta}{2\pi} \qquad x_A = -r_1\theta$$

Differentiate to relate the translational and angular velocities.

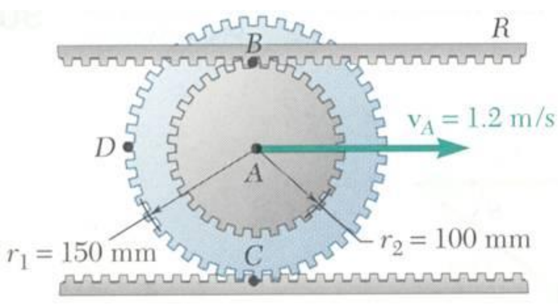
$$v_A = -r_1\omega$$

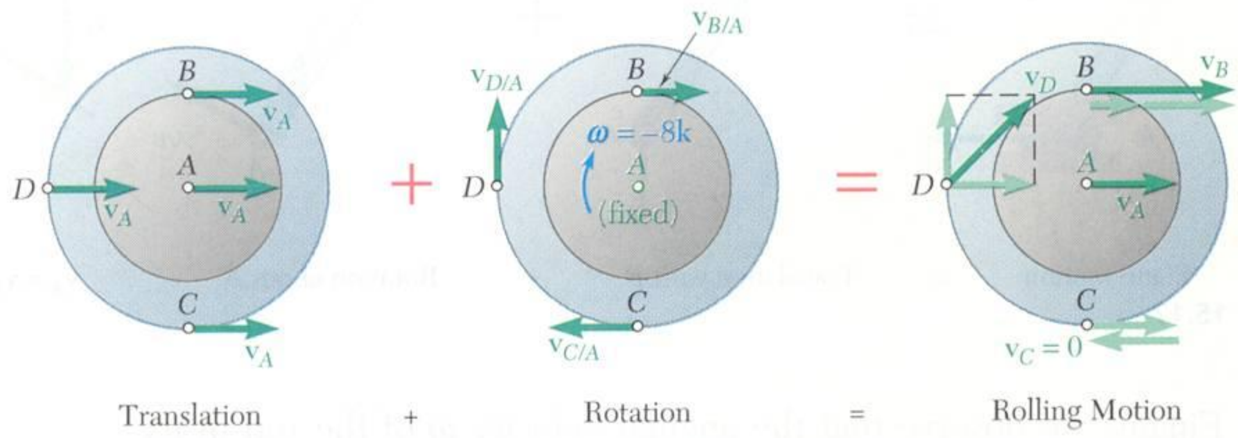
$$\omega = -\frac{v_A}{r_1} = -\frac{1.2\,\mathrm{m/s}}{0.150\,\mathrm{m}}$$

$$\vec{\omega} = \omega\vec{k} = -(8\,\mathrm{rad/s})\vec{k}$$

# Sample Problem 15.2

- For any point $P$ on the gear, $\vec{v}_P = \vec{v}_A + \vec{v}_{P/A} = \vec{v}_A + \omega\vec{k} \times \vec{r}_{P/A}$




Velocity of the upper rack is equal to velocity of point $B$:

$$\vec{v}_R = \vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$

$$= (1.2\,\text{m/s})\vec{i} + (8\,\text{rad/s})\vec{k} \times (0.10\,\text{m})\vec{j}$$

$$= (1.2\,\text{m/s})\vec{i} + (0.8\,\text{m/s})\vec{i}$$

$$\boxed{\vec{v}_R = (2\,\text{m/s})\vec{i}}$$

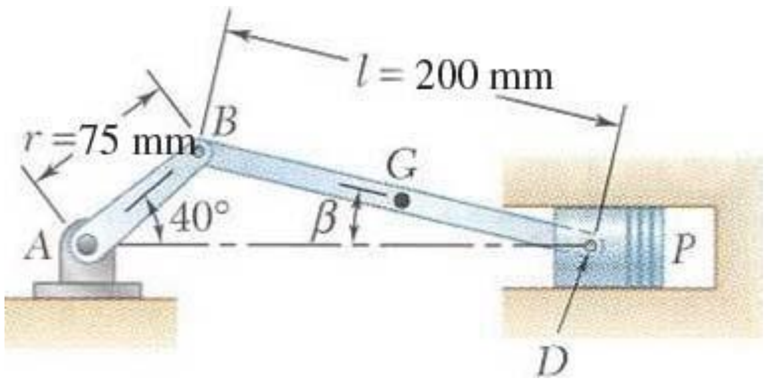
Velocity of the point $D$:

$$\vec{v}_D = \vec{v}_A + \omega\vec{k} \times \vec{r}_{D/A}$$

$$= (1.2\,\text{m/s})\vec{i} + (8\,\text{rad/s})\vec{k} \times (-0.150\,\text{m})\vec{i}$$

$$\boxed{\begin{aligned} \vec{v}_D &= (1.2\,\text{m/s})\vec{i} + (1.2\,\text{m/s})\vec{j} \\ v_D &= 1.697\,\text{m/s} \end{aligned}}$$

## Sample Problem 15.3




The crank *AB* has a constant clockwise angular velocity of 2000 rpm.

For the crank position indicated, determine *(a)* the angular velocity of the connecting rod *BD*, and *(b)* the velocity of the piston *P*.

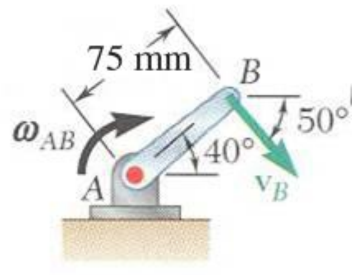
SOLUTION:

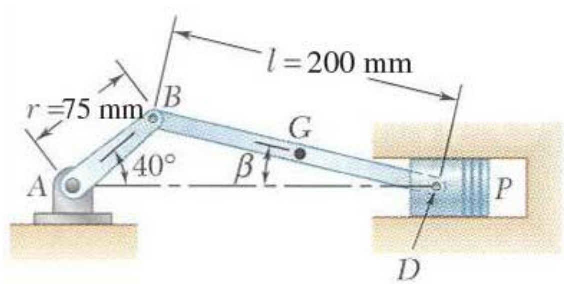
- Will determine the absolute velocity of point *D* with

$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

- The velocity $\vec{v}_B$ is obtained from the given crank rotation data.
- The directions of the absolute velocity $\vec{v}_D$ and the relative velocity $\vec{v}_{D/B}$ are determined from the problem geometry.
- The unknowns in the vector expression are the velocity magnitudes $v_D$ and $v_{D/B}$ which may be determined from the corresponding vector triangle.
- The angular velocity of the connecting rod is calculated from $v_{D/B}$.

# Sample Problem 15.3




<u>SOLUTION</u>:

- Will determine the absolute velocity of point *D* with

$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

- The velocity $\vec{v}_B$ is obtained from the crank rotation data.

$$\omega_{AB} = \left(2000\,\frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\,\text{s}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 209.4\ \text{rad/s}$$

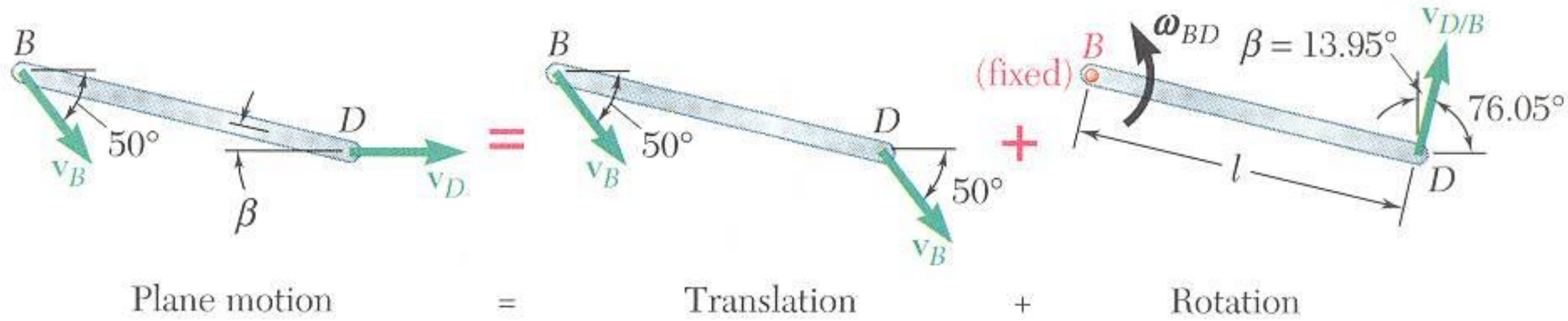
$$v_B = (AB)\,\omega_{AB} = (75\,\text{mm})(209.4\,\text{rad/s}) = 15705\,\text{mm/s}$$

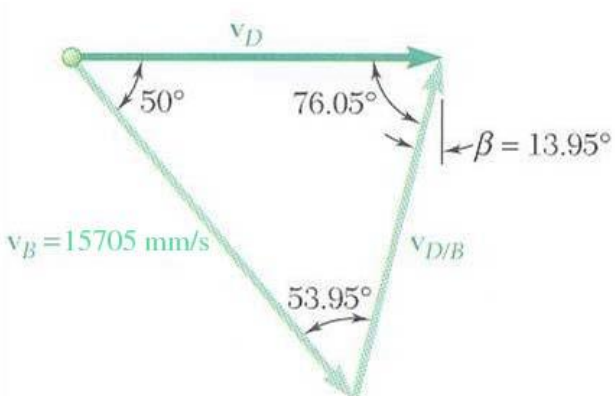
The velocity direction is as shown.

- The direction of the absolute velocity $\vec{v}_D$ is horizontal. The direction of the relative velocity $\vec{v}_{D/B}$ is perpendicular to *BD*. Compute the angle between the horizontal and the connecting rod from the law of sines.

$$\frac{\sin 40°}{200\,\text{mm}} = \frac{\sin \beta}{75\,\text{mm}} \qquad \beta = 13.95°$$

# Sample Problem 15.3




$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

- Determine the velocity magnitudes $v_D$ and $v_{D/B}$ from the vector triangle.

$$\frac{v_D}{\sin 53.95°} = \frac{v_{D/B}}{\sin 50°} = \frac{15705\,\text{mm/s}}{\sin 76.05°}$$

$$v_D = 13083\,\text{mm/s} = 13.08\,\text{m/s}$$

$$v_P = v_D = 13.08\,\text{m/s}$$

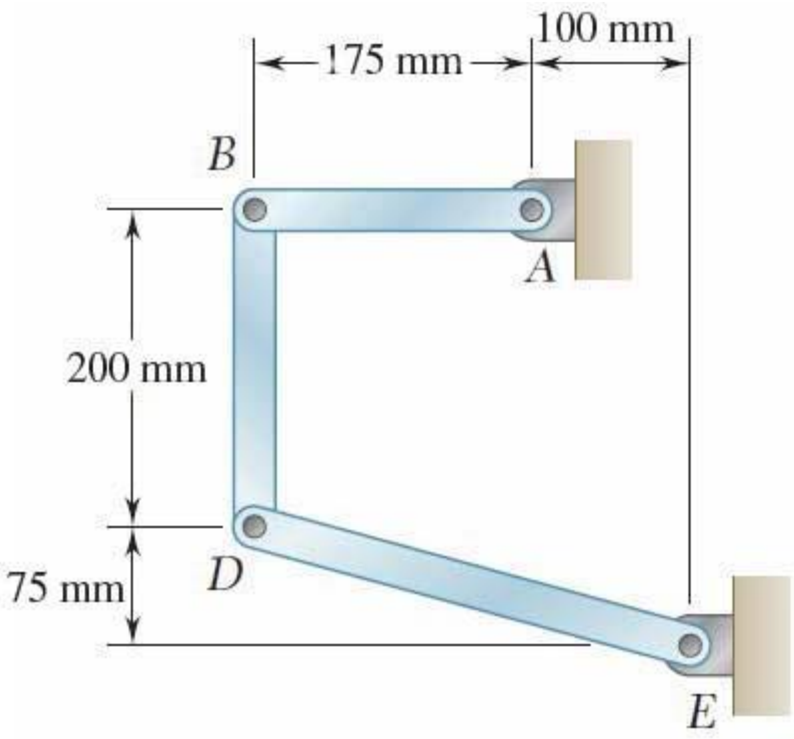
$$v_{D/B} = 12396\,\text{mm/s}$$

$$v_{D/B} = l w_{BD}$$

$$w_{BD} = \frac{v_{D/B}}{l} = \frac{12396\,\text{mm/s}}{200\ \text{mm}} = 62.0\ \text{rad/s}$$

$$\vec{\omega}_{BD} = (62.0\,\text{rad/s})\vec{k}$$

# Group Problem Solving

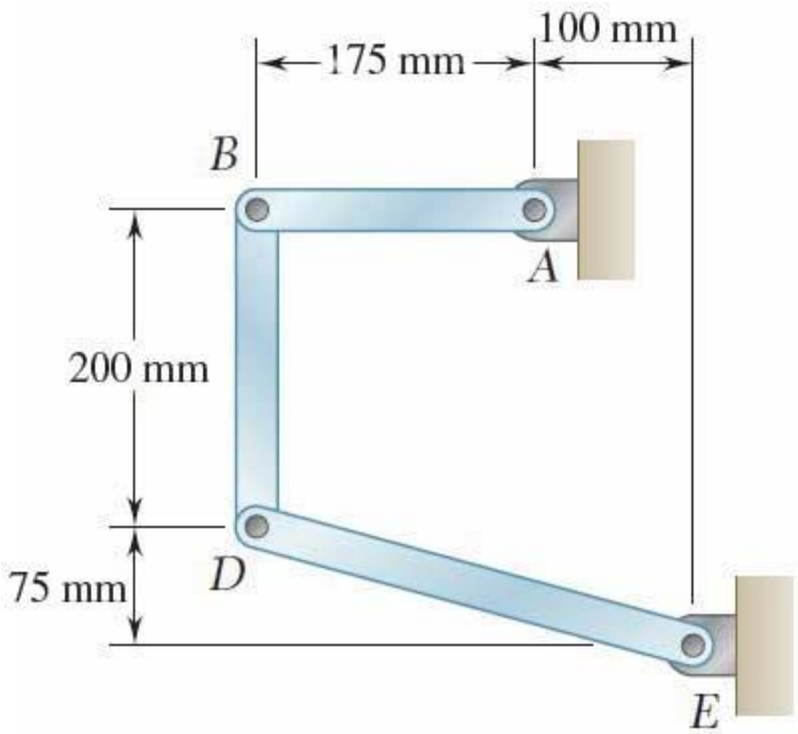



**In the position shown, bar *AB* has an angular velocity of 4 rad/s clockwise. Determine the angular velocity of bars *BD* and *DE*.**

**Which of the following is true?**

**a) The direction of $v_B$ is ↑**
**b) The direction of $v_D$ is →**
**c) Both a) and b) are correct**

# Group Problem Solving




In the position shown, bar *AB* has an angular velocity of 4 rad/s clockwise. Determine the angular velocity of bars *BD* and *DE*.

SOLUTION:

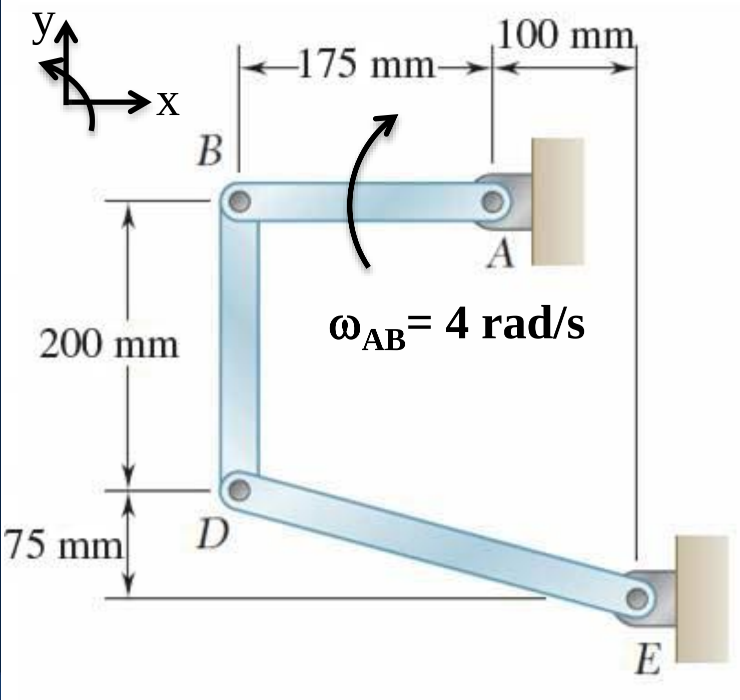
- The displacement of the gear center in one revolution is equal to the outer circumference. Relate the translational and angular displacements. Differentiate to relate the translational and angular velocities.

- The velocity for any point *P* on the gear may be written as

$$\vec{v}_P = \vec{v}_A + \vec{v}_{P/A} = \vec{v}_A + \omega\vec{k} \times \vec{r}_{P/A}$$

Evaluate the velocities of points *B* and *D*.

## Group Problem Solving




**Determine the angular velocity of bars *BD* and *DE*.**

**How should you proceed?**

**Determine $v_B$ with respect to A, then work your way along the linkage to point E.**

**Write $v_B$ in terms of point A, calculate $v_B$.**

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$

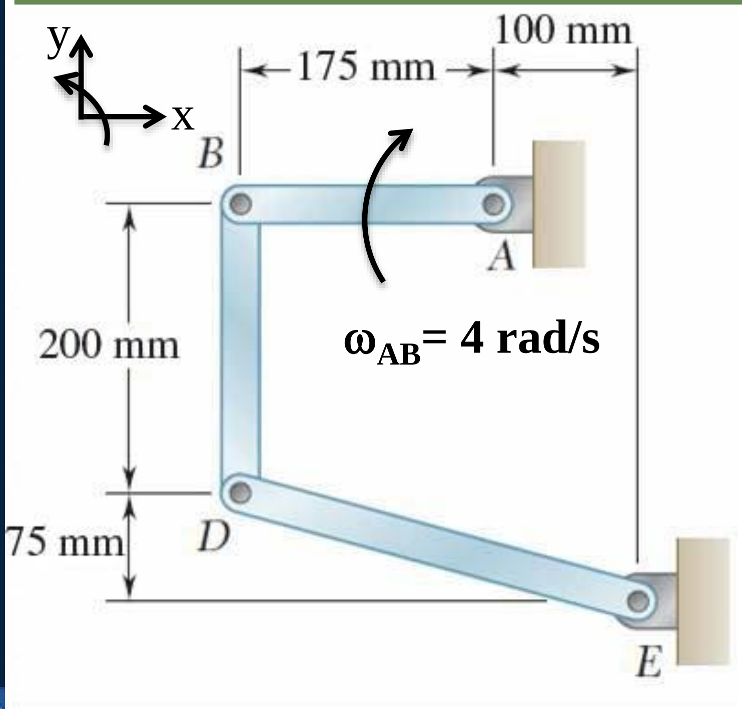
$$\boldsymbol{\omega}_{AB} = -(4\text{ rad/s})\mathbf{k}$$

$$\mathbf{r}_{B/A} = -(175\text{ mm})\mathbf{i} \qquad \mathbf{v}_B = \omega_{AB} \times r_{B/A} = (-4\mathbf{k}) \times (-175\mathbf{i})$$

$$\mathbf{v}_B = (700\text{ mm/s})\mathbf{j}$$

**Does it make sense that $v_B$ is in the +j direction?**

## Group Problem Solving




**Determine $v_D$ with respect to B.**

$$\boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \quad \mathbf{r}_{D/B} = -(200\text{ mm})\mathbf{j}$$

$$\mathbf{v}_D = \mathbf{v}_B + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{D/B} = 700\mathbf{j} + (\omega_{BD}\mathbf{k}) \times (-200\mathbf{j})$$

$$\mathbf{v}_D = 700\mathbf{j} + 200\omega_{BD}\mathbf{i}$$

**Determine $v_D$ with respect to E, then equate it to equation above.**

$$\boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k} \qquad \mathbf{r}_{D/E} = -(275\text{ mm})\mathbf{i} + (75\text{ mm})\mathbf{j}$$

$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = (\omega_{DE}\mathbf{k}) \times (-275\mathbf{i} + 75\mathbf{j})$$

$$\mathbf{v}_D = -275\omega_{DE}\mathbf{j} - 75\omega_{DE}\mathbf{i}$$

**Equating components of the two expressions for $v_D$**

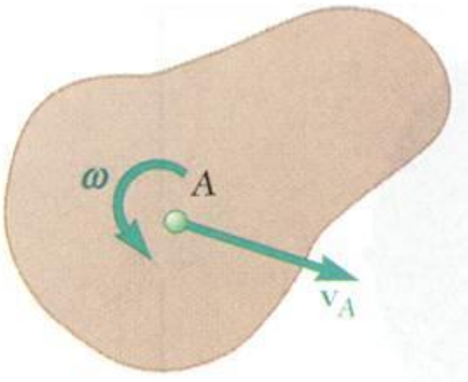
$$\text{j:} \quad 700 = -275\omega_{DE} \qquad \omega_{DE} = -2.5455\text{ rad/s}$$

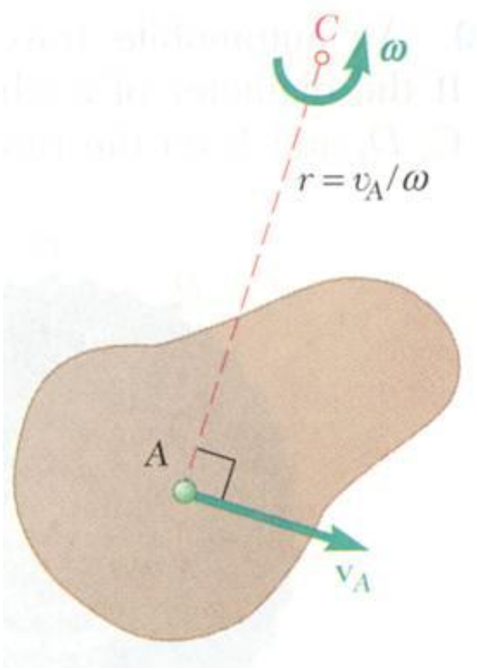
$$\mathbf{i}: \quad 200\omega_{BD} = -75\omega_{DE} \qquad \omega_{BD} = -\frac{3}{8}\omega_{DE}$$

$$\boldsymbol{\omega}_{DE} = 2.55\text{ rad/s} \curvearrowright$$

$$\boldsymbol{\omega}_{BD} = 0.955\text{ rad/s} \curvearrowleft$$

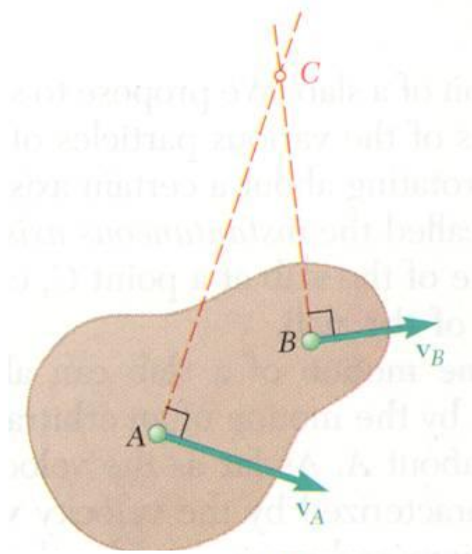
# Instantaneous Center of Rotation in Plane Motion

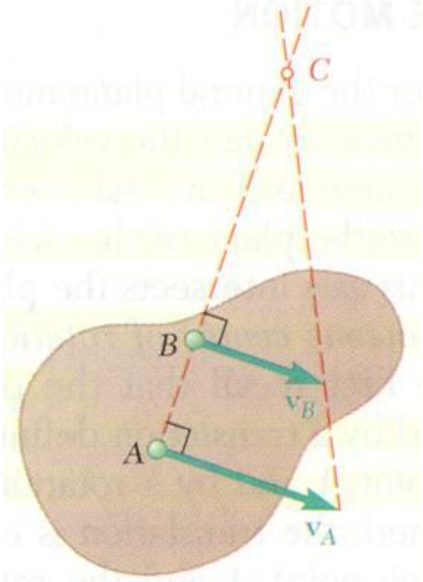



- Plane motion of all particles in a slab can always be replaced by the translation of an arbitrary point *A* and a rotation about *A* with an angular velocity that is independent of the choice of *A*.
- The same translational and rotational velocities at *A* are obtained by allowing the slab to rotate with the same angular velocity about the point *C* on a perpendicular to the velocity at *A*.
- The velocity of all other particles in the slab are the same as originally defined since the angular velocity and translational velocity at *A* are equivalent.
- As far as the velocities are concerned, the slab seems to rotate about the *instantaneous center of rotation C*.

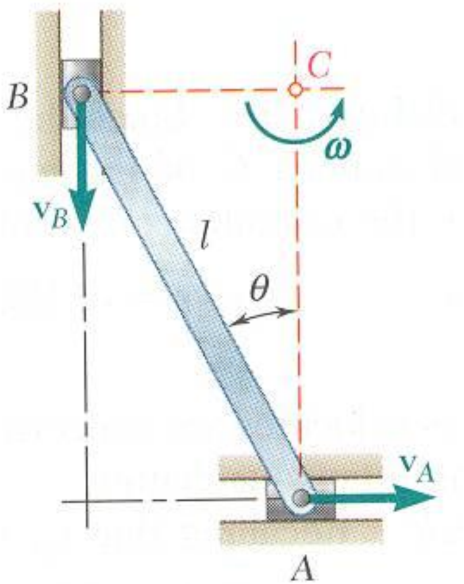
# Instantaneous Center of Rotation in Plane Motion

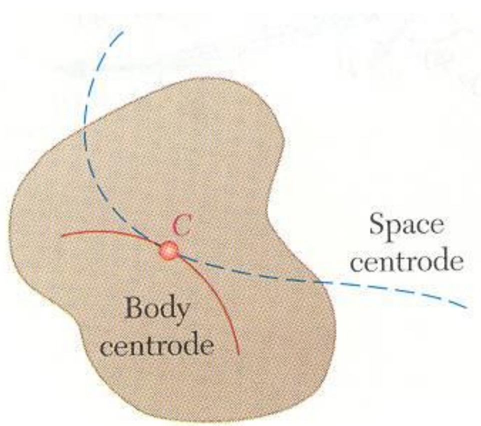



- If the velocity at two points *A* and *B* are known, the instantaneous center of rotation lies at the intersection of the perpendiculars to the velocity vectors through *A* and *B* .
- If the velocity vectors are parallel, the instantaneous center of rotation is at infinity and the angular velocity is zero.
- If the velocity vectors at *A* and *B* are perpendicular to the line *AB*, the instantaneous center of rotation lies at the intersection of the line *AB* with the line joining the extremities of the velocity vectors at *A* and *B*.
- If the velocity magnitudes are equal, the instantaneous center of rotation is at infinity and the angular velocity is zero.

# Instantaneous Center of Rotation in Plane Motion



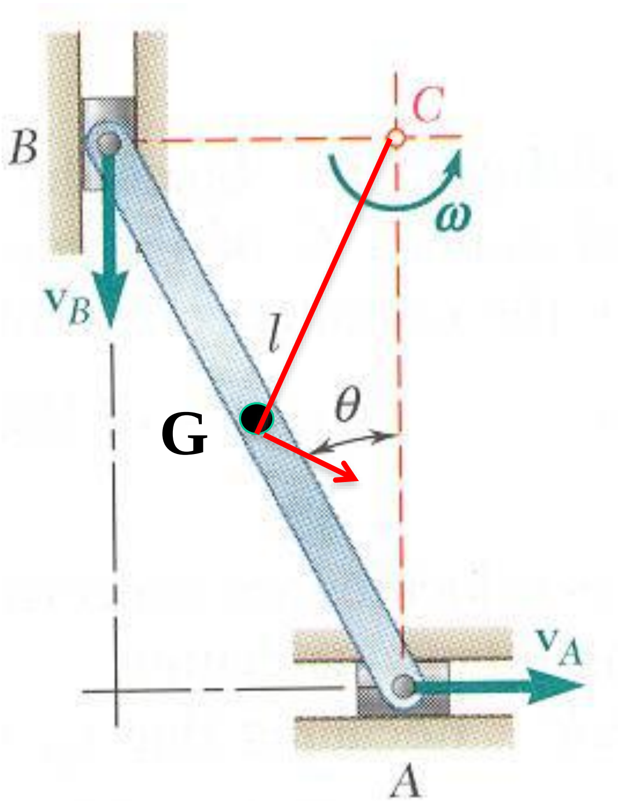




- The instantaneous center of rotation lies at the intersection of the perpendiculars to the velocity vectors through $A$ and $B$ .

$$\omega = \frac{v_A}{AC} = \frac{v_A}{l\cos\theta} \qquad v_B = (BC)\omega = (l\sin\theta)\frac{v_A}{l\cos\theta} = v_A\tan\theta$$

- The velocities of all particles on the rod are as if they were rotated about $C$.
- The particle at the center of rotation has zero velocity.
- The particle coinciding with the center of rotation changes with time and the acceleration of the particle at the instantaneous center of rotation is not zero.
- The acceleration of the particles in the slab cannot be determined as if the slab were simply rotating about $C$.
- The trace of the locus of the center of rotation on the body is the body centrode and in space is the space centrode.

# Instantaneous Center of Rotation in Plane Motion




**At the instant shown, what is the approximate direction of the velocity of point G, the center of bar AB?**

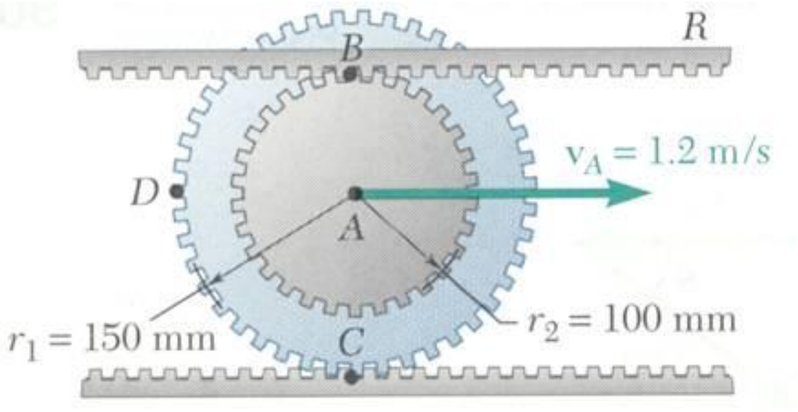
a) →

b) ↗

c) ↘

d) ↓

## Sample Problem 15.4



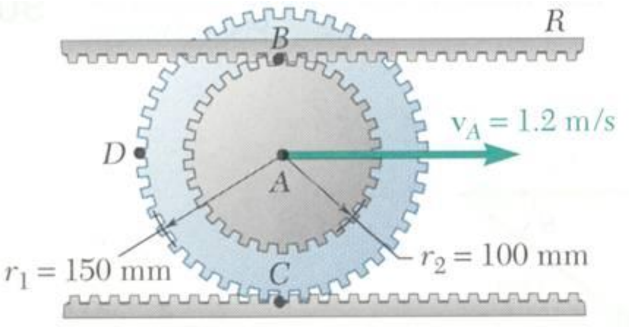

The double gear rolls on the stationary lower rack: the velocity of its center is 1.2 m/s.

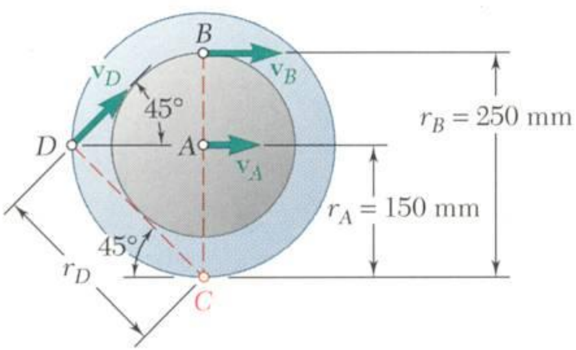
Determine *(a)* the angular velocity of the gear, and *(b)* the velocities of the upper rack *R* and point *D* of the gear.

SOLUTION:

- The point *C* is in contact with the stationary lower rack and, instantaneously, has zero velocity. It must be the location of the instantaneous center of rotation.
- Determine the angular velocity about *C* based on the given velocity at *A*.
- Evaluate the velocities at *B* and *D* based on their rotation about *C*.

# Sample Problem 15.4




SOLUTION:

- The point $C$ is in contact with the stationary lower rack and, instantaneously, has zero velocity. It must be the location of the instantaneous center of rotation.
- Determine the angular velocity about $C$ based on the given velocity at $A$.

$$v_A = r_A \omega \qquad \omega = \frac{v_A}{r_A} = \frac{1.2\,\text{m/s}}{0.15\,\text{m}} = 8\,\text{rad/s}$$

- Evaluate the velocities at $B$ and $D$ based on their rotation about $C$.

$$v_R = v_B = r_B \omega = (0.25\,\text{m})(8\,\text{rad/s})$$

$$\vec{v}_R = (2\,\text{m/s})\vec{i}$$

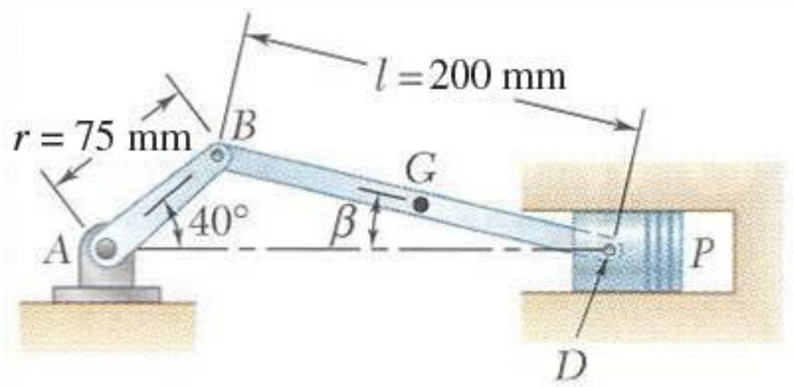
$$r_D = (0.15\,\text{m})\sqrt{2} = 0.2121\,\text{m}$$

$$v_D = r_D \omega = (0.2121\,\text{m})(8\,\text{rad/s})$$

$$v_D = 1.697\,\text{m/s}$$

$$\vec{v}_D = \left(1.2\vec{i} + 1.2\vec{j}\right)(\text{m/s})$$

## Sample Problem 15.5



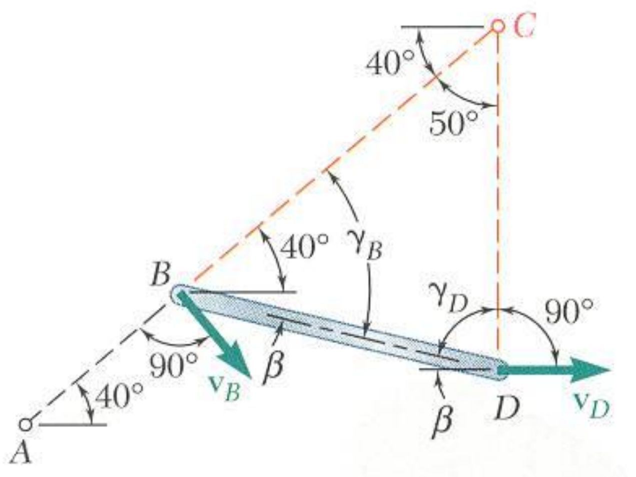

The crank *AB* has a constant clockwise angular velocity of 2000 rpm.

For the crank position indicated, determine *(a)* the angular velocity of the connecting rod *BD*, and *(b)* the velocity of the piston *P*.

SOLUTION:

- Determine the velocity at *B* from the given crank rotation data.
- The direction of the velocity vectors at *B* and *D* are known. The instantaneous center of rotation is at the intersection of the perpendiculars to the velocities through *B* and *D*.
- Determine the angular velocity about the center of rotation based on the velocity at *B*.
- Calculate the velocity at *D* based on its rotation about the instantaneous center of rotation.

## Sample Problem 15.5




$$\gamma_B = 40° + \beta = 53.95°$$
$$\gamma_D = 90° - \beta = 76.05°$$

$$\frac{BC}{\sin 76.05°} = \frac{CD}{\sin 53.95°} = \frac{200 \text{ mm}}{\sin 50°}$$

$$BC = 253.4 \text{ mm} \quad CD = 211.1 \text{ mm}$$

SOLUTION:

- From Sample Problem 15.3,

$$\vec{v}_B = \left(10095\vec{i} - 12031\vec{j}\right)(\text{mm/s}) \qquad v_B = 15705 \text{ mm/s}$$
$$b = 13.95°$$

- The instantaneous center of rotation is at the intersection of the perpendiculars to the velocities through *B* and *D*.

- Determine the angular velocity about the center of rotation based on the velocity at *B*.

$$v_B = (BC)w_{BD}$$
$$w_{BD} = \frac{v_B}{BC} = \frac{15705 \text{ mm/s}}{253.4 \text{ mm}}$$

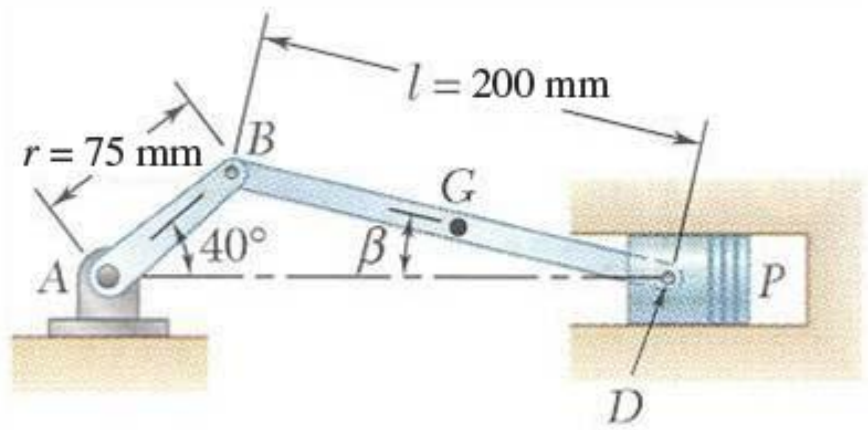
$$\omega_{BD} = 62.0 \text{ rad/s}$$

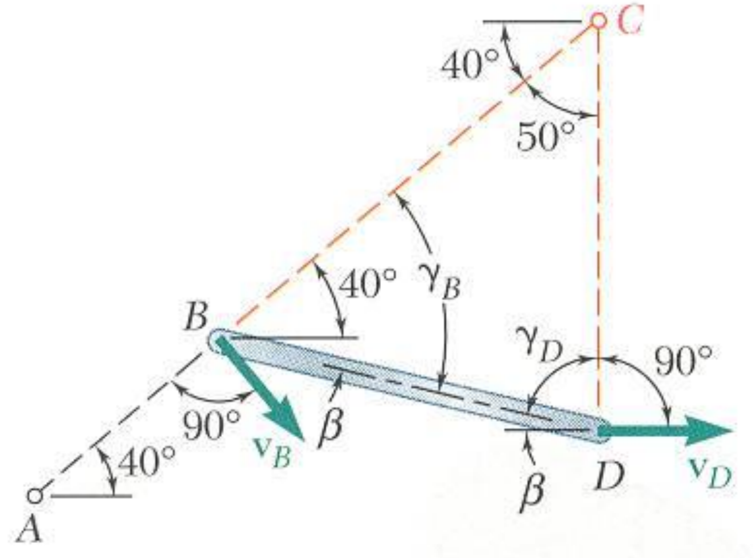
- Calculate the velocity at *D* based on its rotation about the instantaneous center of rotation.

$$v_D = (CD)w_{BD} = (211.1 \text{ mm})(62.0 \text{ rad/s})$$

$$v_P = v_D = 13090 \text{ mm/s} = 1.309 \text{ m/s}$$

## Instantaneous Center of Zero Velocity

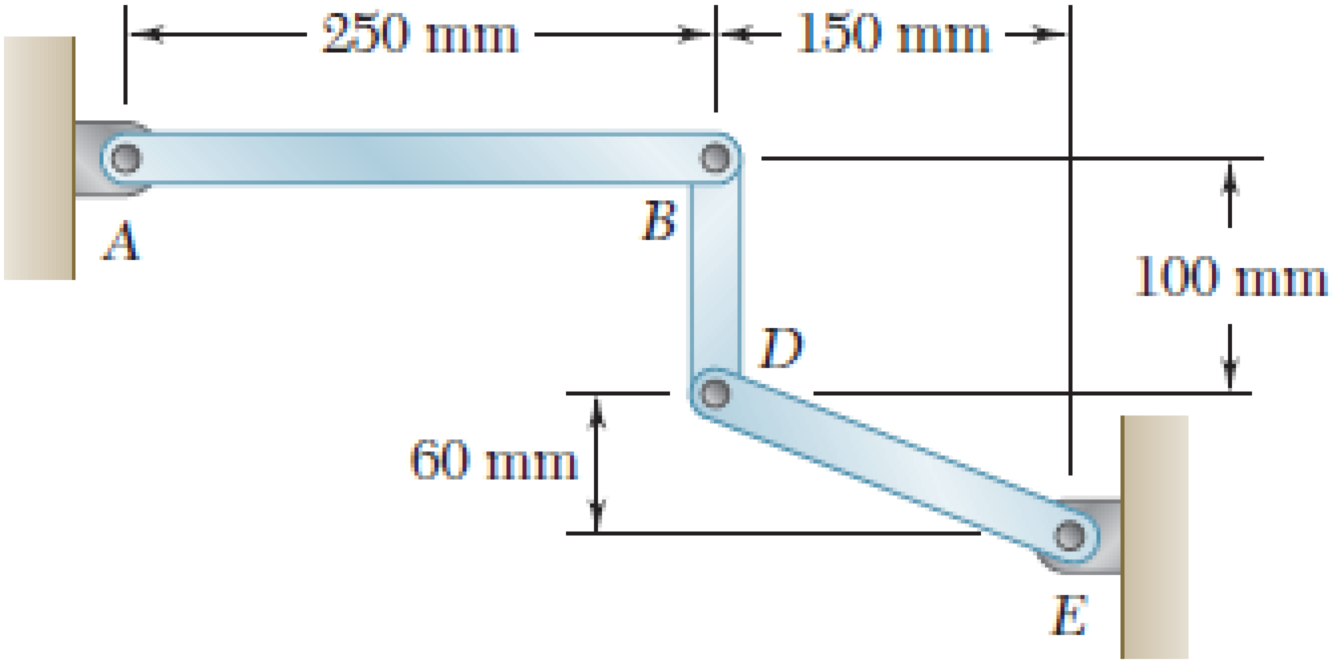



**What happens to the location of the instantaneous center of velocity if the crankshaft angular velocity increases from 2000 rpm in the previous problem to 3000 rpm?**

**What happens to the location of the instantaneous center of velocity if the angle β is 0?**

## Group Problem Solving




**In the position shown, bar *AB* has an angular velocity of 4 rad/s clockwise. Determine the angular velocity of bars *BD* and *DE*.**

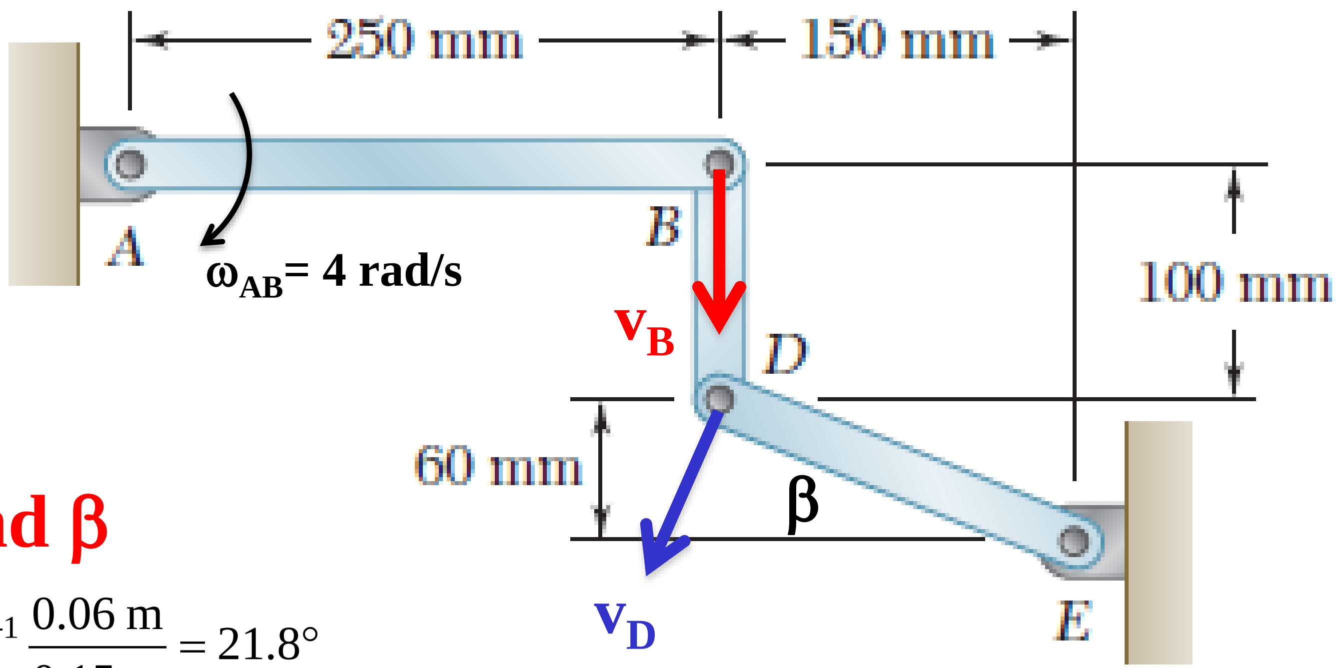
## Group Problem Solving

**What is the velocity of B?**

$$\mathbf{v}_B = (AB)\omega_{AB} = (0.25\text{ m})(4\text{ rad/s}) = 1\text{ m/s}$$

**What direction is the velocity of B?**

**What direction is the velocity of D?**

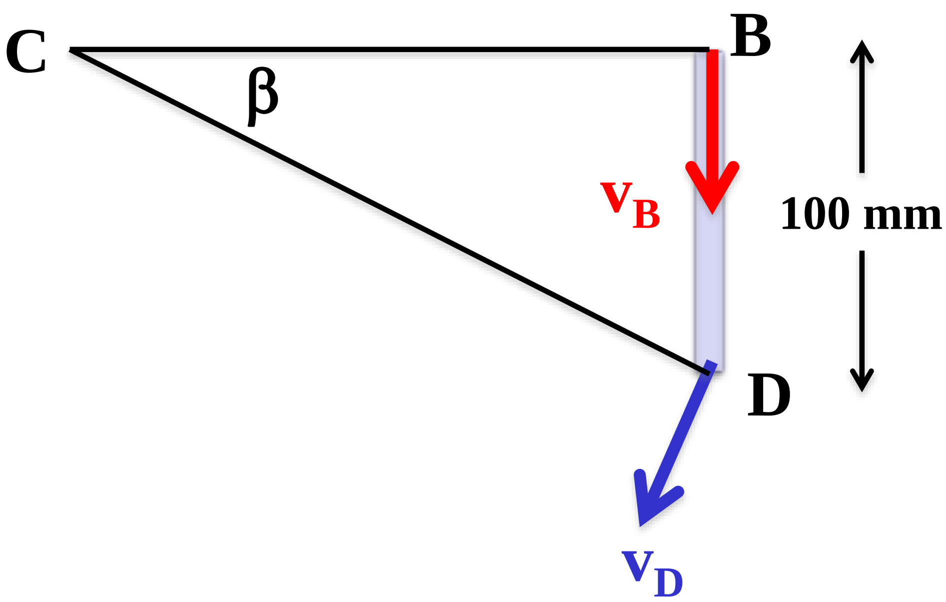



**Find β**

$$\beta = \tan^{-1}\frac{0.06\text{ m}}{0.15\text{ m}} = 21.8°$$

# Group Problem Solving

**Locate instantaneous center *C* at intersection of lines drawn perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_D$.**




**Find distances BC and DC**

$$BC = \frac{0.1\text{ m}}{\tan\beta} = \frac{0.1\text{ m}}{\tan 21.8°} = 0.25\text{ m}$$

$$DC = \frac{0.25\text{ m}}{\cos\beta} = \frac{0.25\text{ m}}{\cos 21.8°} = 0.2693\text{ m}$$

**Calculate $\omega_{BD}$**

$$v_B = (BC)\omega_{BD}$$

$$1\text{ m/s} = (0.25\text{ m})\omega_{BD}$$

$$\boldsymbol{\omega}_{BD} = 4\text{ rad/s} \circlearrowright$$

**Find $\omega_{DE}$**

$$v_D = (DC)\omega_{BD} = \frac{0.25\text{ m}}{\cos\beta}(4\text{ rad/s})$$

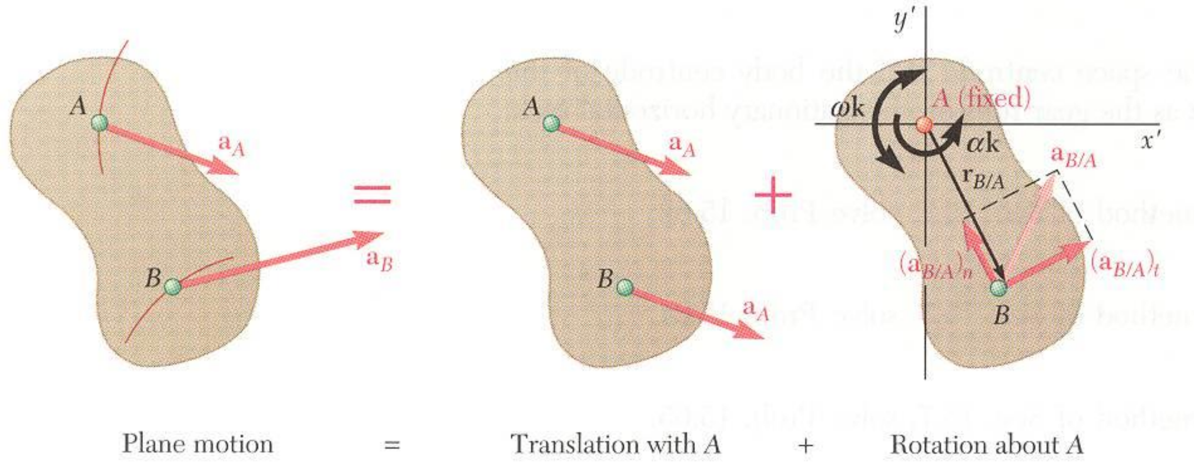
$$v_D = (DE)\omega_{DE}; \quad \frac{1\text{ m/s}}{\cos\beta} = \frac{0.15\text{ m}}{\cos\beta}\omega_{DE};$$

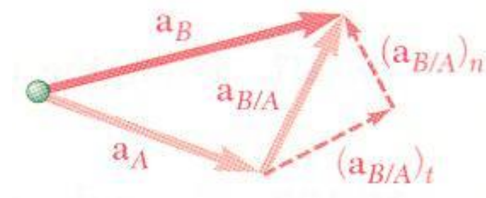
$$\boldsymbol{\omega}_{DE} = 6.67\text{ rad/s}$$

## Absolute and Relative Acceleration in Plane Motion

**As the bicycle accelerates, a point on the top of the wheel will have acceleration due to the acceleration from the axle (the overall linear acceleration of the bike), the tangential acceleration of the wheel from the angular acceleration, and the normal acceleration due to the angular velocity.**

# Absolute and Relative Acceleration in Plane Motion




- Absolute acceleration of a particle of the slab,

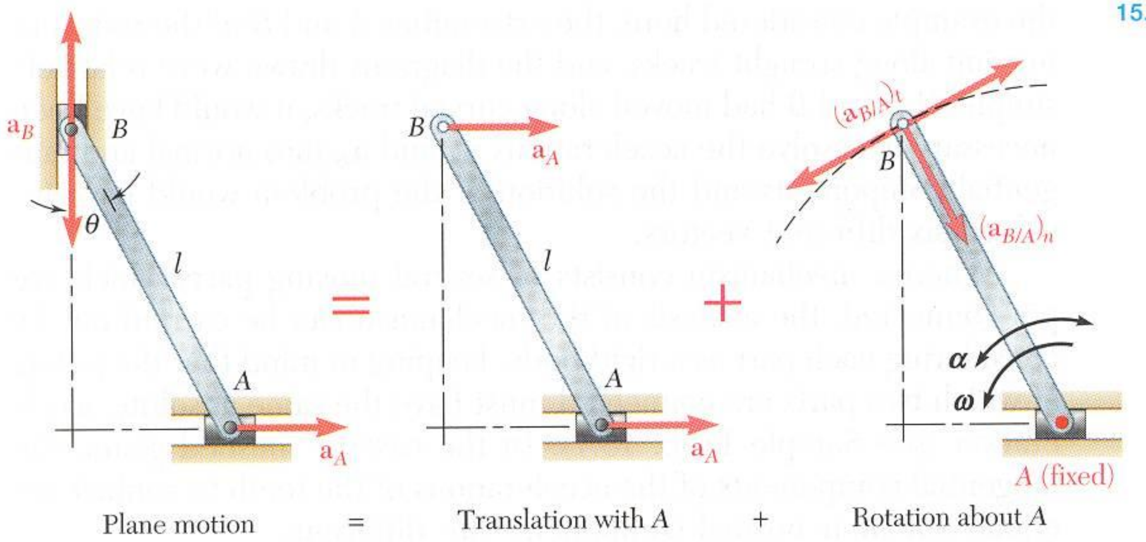
$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

- Relative acceleration $\vec{a}_{B/A}$ associated with rotation about $A$ includes tangential and normal components,

$$\left(\vec{a}_{B/A}\right)_t = \alpha\vec{k} \times \vec{r}_{B/A} \qquad \left(a_{B/A}\right)_t = r\alpha$$

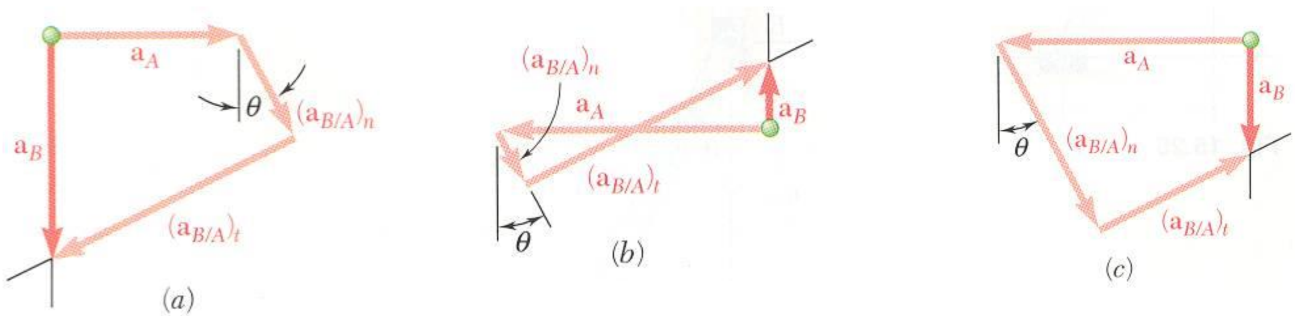
$$\left(\vec{a}_{B/A}\right)_n = -\omega^2 \vec{r}_{B/A} \qquad \left(a_{B/A}\right)_n = r\omega^2$$

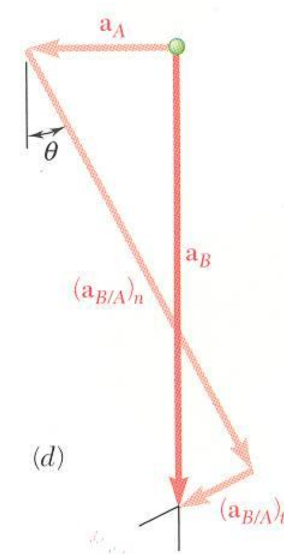
# Absolute and Relative Acceleration in Plane Motion




- Given $\vec{a}_A$ and $\vec{v}_A$, determine $\vec{a}_B$ and $\vec{\alpha}$.

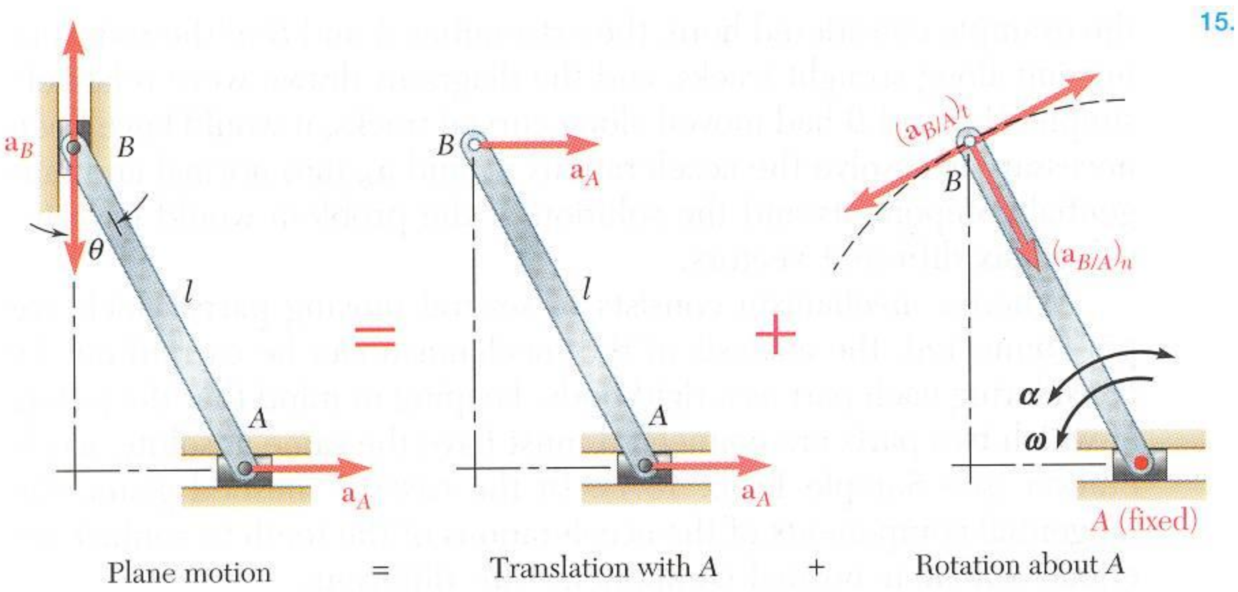
$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$
$$= \vec{a}_A + \left(\vec{a}_{B/A}\right)_n + \left(\vec{a}_{B/A}\right)_t$$

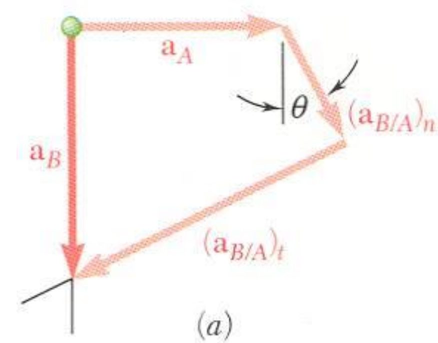



- Vector result depends on sense of $\vec{a}_A$ and the relative magnitudes of $a_A$ and $\left(a_{B/A}\right)_n$
- Must also know angular velocity $\omega$.

# Absolute and Relative Acceleration in Plane Motion



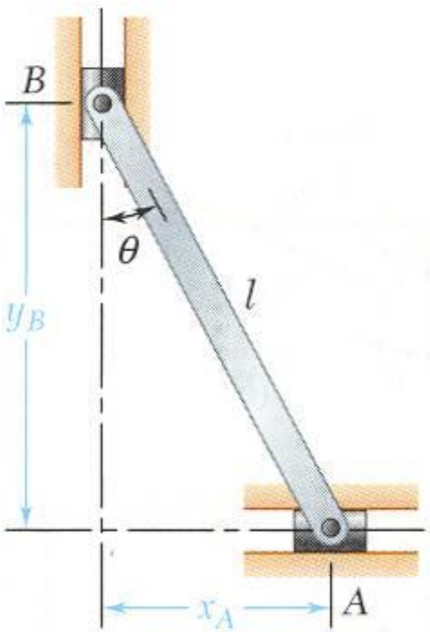

- Write $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ in terms of the two component equations,

$\xrightarrow{+}$ $x$ components: $$0 = a_A + l\omega^2 \sin\theta - l\alpha\cos\theta$$

$+\uparrow$ $y$ components: $$-a_B = -l\omega^2\cos\theta - l\alpha\sin\theta$$

- Solve for $a_B$ and $\alpha$.

## Analysis of Plane Motion in Terms of a Parameter




- In some cases, it is advantageous to determine the absolute velocity and acceleration of a mechanism directly.

$$x_A = l\sin\theta \qquad y_B = l\cos\theta$$

$$v_A = \dot{x}_A = l\dot{\theta}\cos\theta = l\omega\cos\theta \qquad v_B = \dot{y}_B = -l\dot{\theta}\sin\theta = -l\omega\sin\theta$$

$$a_A = \ddot{x}_A = -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta = -l\omega^2\sin\theta + l\alpha\cos\theta$$

$$a_B = \ddot{y}_B = -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta = -l\omega^2\cos\theta - l\alpha\sin\theta$$

## Concept Question

**You have made it to the kickball championship game. As you try to kick home the winning run, your mind naturally drifts towards dynamics. Which of your following thoughts is TRUE, and causes you to shank the ball horribly straight to the pitcher?**

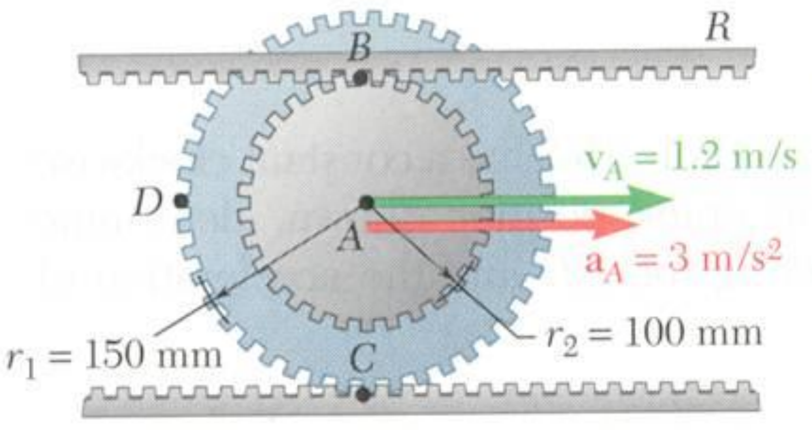
**A) Energy will not be conserved when I kick this ball**

**B) In general, the linear acceleration of my knee is equal to the linear acceleration of my foot**

**C) Throughout the kick, my foot will only have tangential acceleration.**

**D) In general, the angular velocity of the upper leg (thigh) will be the same as the angular velocity of the lower leg**

# Sample Problem 15.6



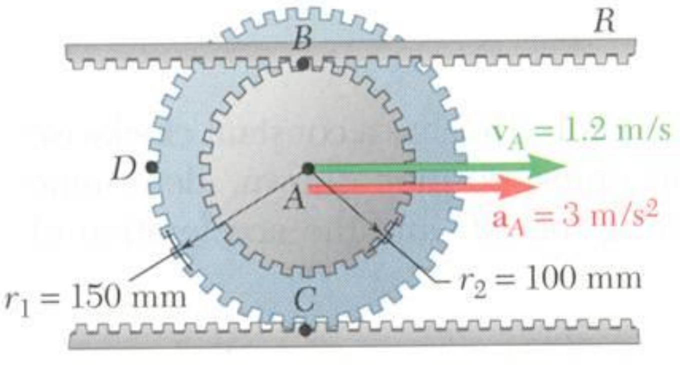

The center of the double gear has a velocity and acceleration to the right of 1.2 m/s and 3 m/s$^2$, respectively. The lower rack is stationary.

Determine *(a)* the angular acceleration of the gear, and *(b)* the acceleration of points *B*, *C*, and *D*.

SOLUTION:

- The expression of the gear position as a function of $\theta$ is differentiated twice to define the relationship between the translational and angular accelerations.
- The acceleration of each point on the gear is obtained by adding the acceleration of the gear center and the relative accelerations with respect to the center. The latter includes normal and tangential acceleration components.

## Sample Problem 15.6




SOLUTION:

- The expression of the gear position as a function of $\theta$ is differentiated twice to define the relationship between the translational and angular accelerations.

$$x_A = -r_1\theta$$

$$v_A = -r_1\dot{\theta} = -r_1\omega$$

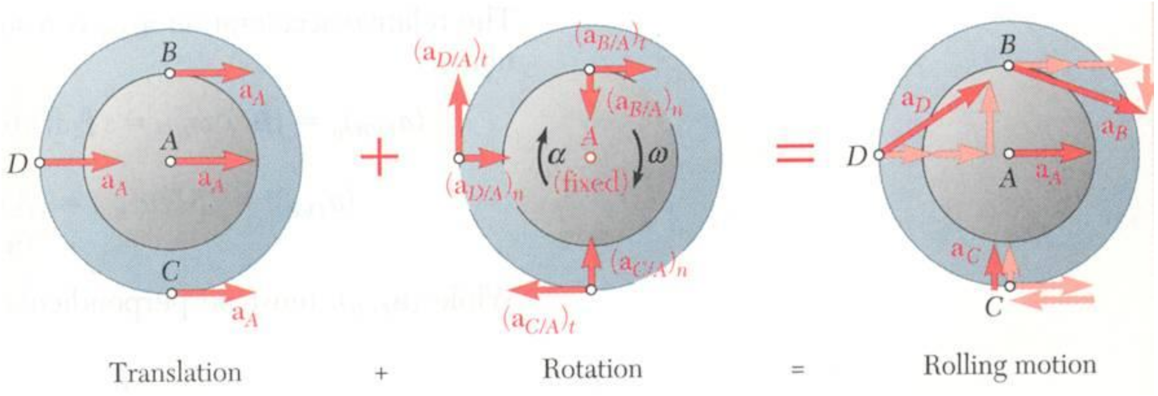
$$\omega = -\frac{v_A}{r_1} = -\frac{1.2\,\text{m/s}}{0.150\text{ m}} = -8\,\text{rad/s}$$

$$a_A = -r_1\ddot{\theta} = -r_1\alpha$$

$$\alpha = -\frac{a_A}{r_1} = -\frac{3\,\text{m/s}^2}{0.150\text{ m}}$$

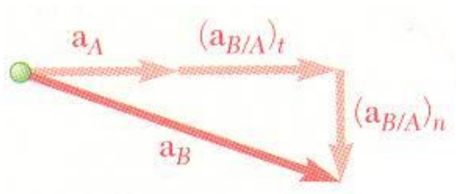
$$\vec{\alpha} = \alpha\vec{k} = -\left(20\,\text{rad/s}^2\right)\vec{k}$$

## Sample Problem 15.6




- The acceleration of each point is obtained by adding the acceleration of the gear center and the relative accelerations with respect to the center.

  The latter includes normal and tangential acceleration components.




$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} = \vec{a}_A + \left(\vec{a}_{B/A}\right)_t + \left(\vec{a}_{B/A}\right)_n$$

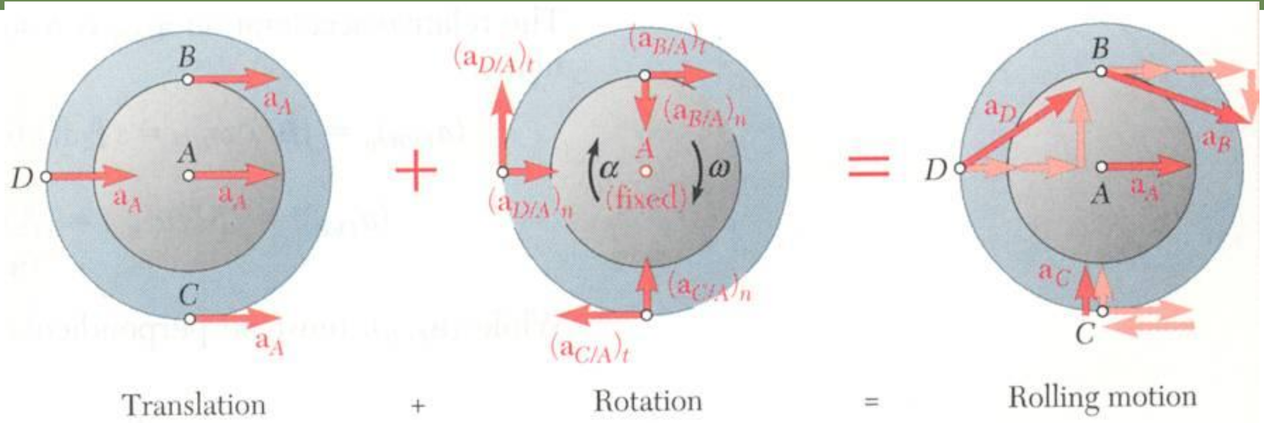
$$= \vec{a}_A + \alpha\vec{k} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$$

$$= \left(3\,\mathrm{m/s^2}\right)\vec{i} - \left(20\,\mathrm{rad/s^2}\right)\vec{k} \times (0.100\,\mathrm{m})\vec{j} - (8\,\mathrm{rad/s})^2(-0.100\,\mathrm{m})\vec{j}$$

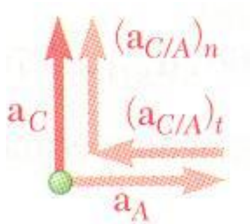
$$= \left(3\,\mathrm{m/s^2}\right)\vec{i} + \left(2\,\mathrm{m/s^2}\right)\vec{i} - \left(6.40\,\mathrm{m/s^2}\right)\vec{j}$$

$$\vec{a}_B = \left(5\,\mathrm{m}/s^2\right)\vec{i} - \left(6.40\,\mathrm{m/s^2}\right)\vec{j} \qquad a_B = 8.12\,\mathrm{m/s^2}$$

# Sample Problem 15.6




$$\vec{a}_C = \vec{a}_A + \vec{a}_{C/A} = \vec{a}_A + \alpha\vec{k}\times\vec{r}_{C/A} - \omega^2\vec{r}_{C/A}$$

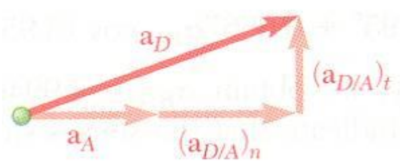



$$= \left(3\,\mathrm{m/s^2}\right)\vec{i} - \left(20\,\mathrm{rad/s^2}\right)\vec{k}\times(-0.150\,\mathrm{m})\vec{j} - (8\,\mathrm{rad/s})^2(-0.150\,\mathrm{m})\vec{j}$$

$$= \left(3\,\mathrm{m/s^2}\right)\vec{i} - \left(3\,\mathrm{m/s^2}\right)\vec{i} + \left(9.60\,\mathrm{m/s^2}\right)\vec{j}$$

$$\boxed{\vec{a}_c = \left(9.60\,\mathrm{m/s^2}\right)\vec{j}}$$

$$\vec{a}_D = \vec{a}_A + \vec{a}_{D/A} = \vec{a}_A + \alpha\vec{k}\times\vec{r}_{D/A} - \omega^2\vec{r}_{D/A}$$

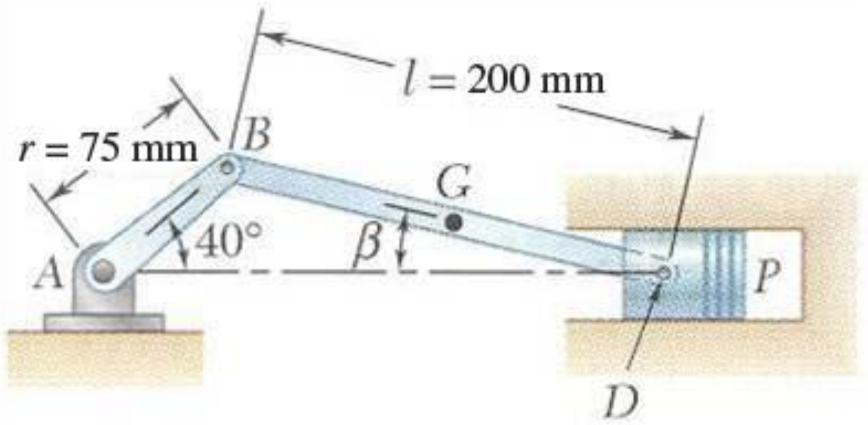



$$= \left(3\,\mathrm{m/s^2}\right)\vec{i} - \left(20\,\mathrm{rad/s^2}\right)\vec{k}\times(-0.150\,\mathrm{m})\vec{i} - (8\,\mathrm{rad/s})^2(-0.150\mathrm{m})\vec{i}$$

$$= \left(3\,\mathrm{m/s^2}\right)\vec{i} + \left(3\,\mathrm{m/s^2}\right)\vec{j} + \left(9.60\,\mathrm{m/s^2}\right)\vec{i}$$

$$\boxed{\vec{a}_D = \left(12.6\,\mathrm{m/s^2}\right)\vec{i} + \left(3\,\mathrm{m/s^2}\right)\vec{j} \qquad a_D = 12.95\,\mathrm{m/s^2}}$$

# Sample Problem 15.7




Crank *AB* of the engine system has a constant clockwise angular velocity of 2000 rpm.

For the crank position shown, determine the angular acceleration of the connecting rod *BD* and the acceleration of point *D*.

SOLUTION:

- The angular acceleration of the connecting rod *BD* and the acceleration of point *D* will be determined from

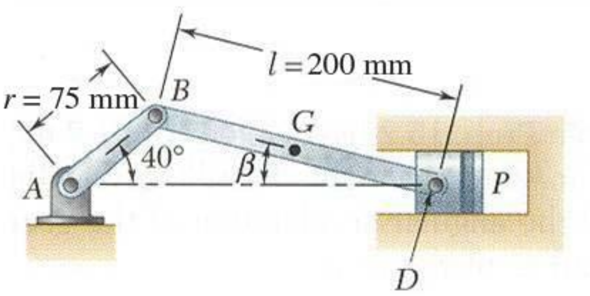
$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B} = \vec{a}_B + \left(\vec{a}_{D/B}\right)_t + \left(\vec{a}_{D/B}\right)_n$$

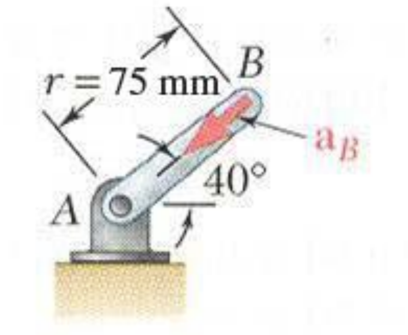
- The acceleration of *B* is determined from the given rotation speed of *AB*.

- The directions of the accelerations $\vec{a}_D, \left(\vec{a}_{D/B}\right)_t$, and $\left(\vec{a}_{D/B}\right)_n$ are determined from the geometry.

- Component equations for acceleration of point *D* are solved simultaneously for acceleration of *D* and angular acceleration of the connecting rod.

# Sample Problem 15.7




SOLUTION:

- The angular acceleration of the connecting rod *BD* and the acceleration of point *D* will be determined from

$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B} = \vec{a}_B + \left(\vec{a}_{D/B}\right)_t + \left(\vec{a}_{D/B}\right)_n$$

- The acceleration of *B* is determined from the given rotation speed of *AB*.

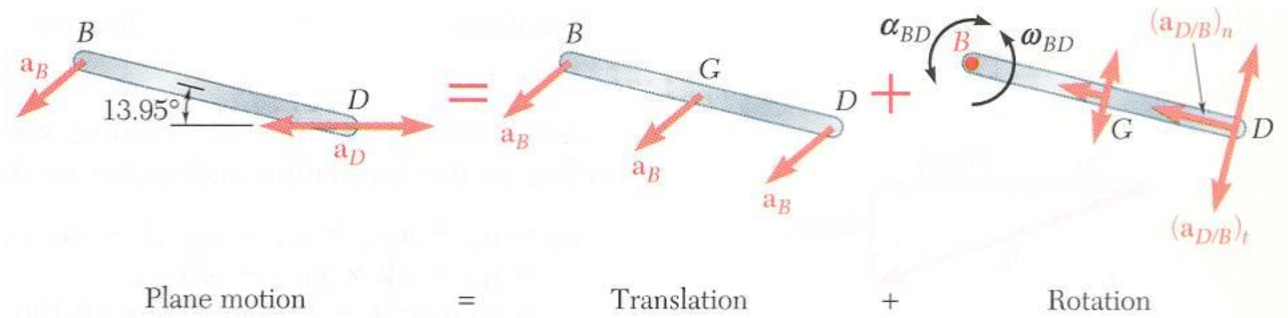
$$\omega_{AB} = 2000\,\text{rpm} = 209.4\,\text{rad/s} = \text{constant}$$

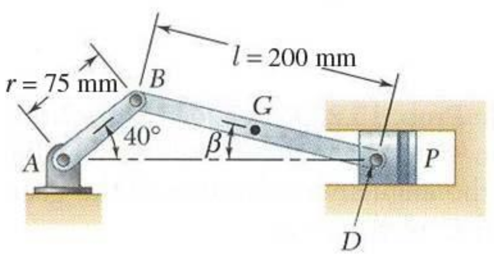
$$\alpha_{AB} = 0$$

$$a_B = r\omega_{AB}^2 = \left(\tfrac{75}{100}\,\text{m}\right)\left(209.4\,\text{rad/s}\right)^2 = 3289\,\text{m/s}^2$$

$$\vec{a}_B = \left(3289\,\text{m/s}^2\right)\left(-\cos 40°\vec{i} - \sin 40°\vec{j}\right)$$

## Sample Problem 15.7




- The directions of the accelerations $\vec{a}_D, \left(\vec{a}_{D/B}\right)_t$, and $\left(\vec{a}_{D/B}\right)_n$ are determined from the geometry.

$$\vec{a}_D = \mp a_D \vec{i}$$

From Sample Problem 15.3, $\omega_{BD} = 62.0$ rad/s, $\beta = 13.95°$.

$$\left(a_{D/B}\right)_n = (BD)\omega_{BD}^2 = \left(\tfrac{200}{1000}\,\text{m}\right)(62.0\,\text{rad/s})^2 = 768.8\,\text{m/s}^2$$

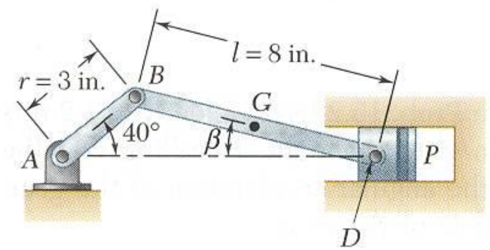
$$\left(\vec{a}_{D/B}\right)_n = \left(768.8\,\text{m/s}^2\right)\left(-\cos 13.95°\vec{i} + \sin 13.95°\vec{j}\right)$$

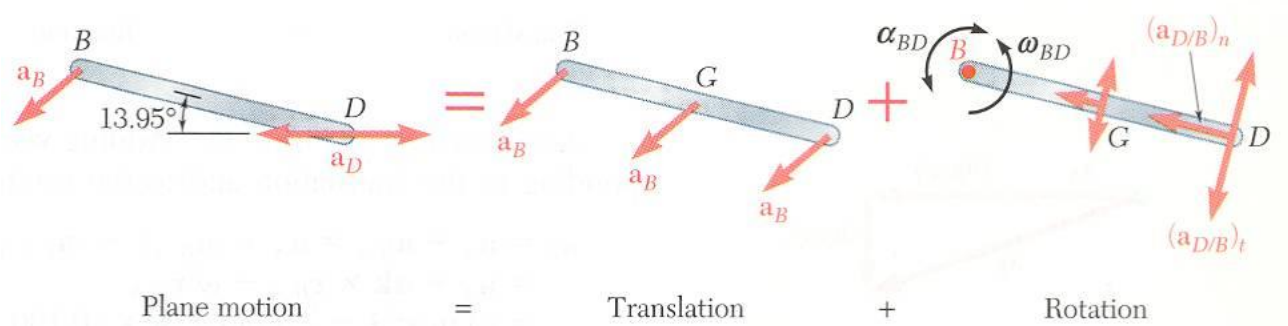
$$\left(a_{D/B}\right)_t = (BD)\alpha_{BD} = \left(\tfrac{200}{1000}\,\text{m}\right)\alpha_{BD} = 0.2\alpha_{BD}$$

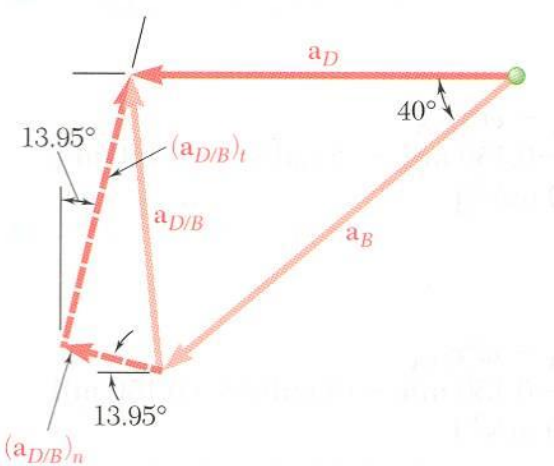
The direction of $(a_{D/B})_t$ is known but the sense is not known,

$$\left(\vec{a}_{D/B}\right)_t = \left(0.2\alpha_{BD}\right)\left(\pm\sin 76.05°\vec{i} \pm \cos 76.05°\vec{j}\right)$$

# Sample Problem 15.7




- Component equations for acceleration of point *D* are solved simultaneously.

$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B} = \vec{a}_B + \left(\vec{a}_{D/B}\right)_t + \left(\vec{a}_{D/B}\right)_n$$

*x* components:

$$-a_D = -3289\cos 40° - 768.8\cos 13.95° + 0.2a_{BD}\sin 13.95°$$

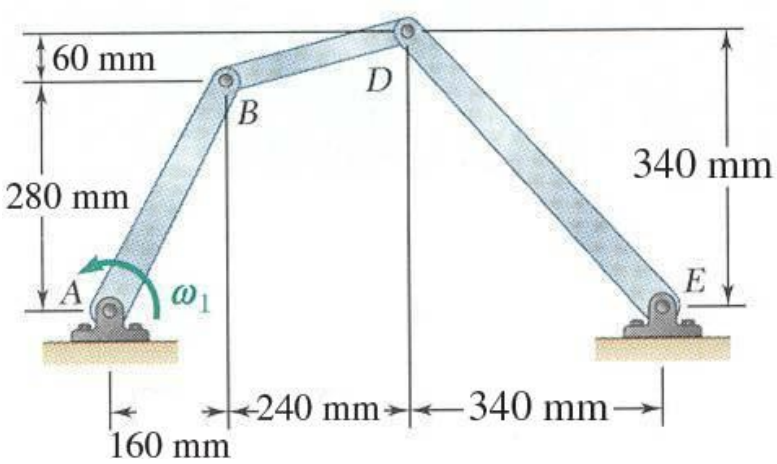
*y* components:

$$0 = -3289\sin 40° + 768.8\sin 13.95° + 0.2a_{BD}\cos 13.95°$$

$$\vec{a}_{BD} = \left(9940\ \mathrm{rad/s^2}\right)\vec{k}$$

$$\vec{a}_D = -\left(2790\ \mathrm{m/s^2}\right)\vec{i}$$

## Sample Problem 15.8




In the position shown, crank *AB* has a constant angular velocity $\omega_1$ = 20 rad/s counterclockwise.

Determine the angular velocities and angular accelerations of the connecting rod *BD* and crank *DE*.

SOLUTION:

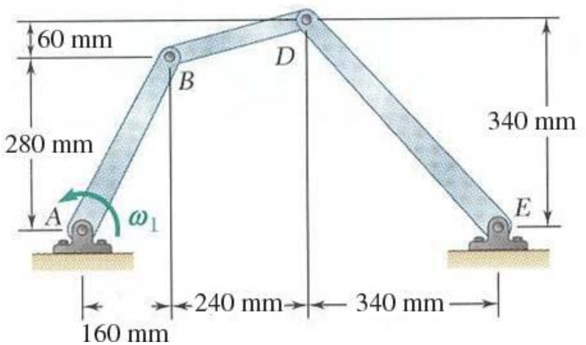
- The angular velocities are determined by simultaneously solving the component equations for

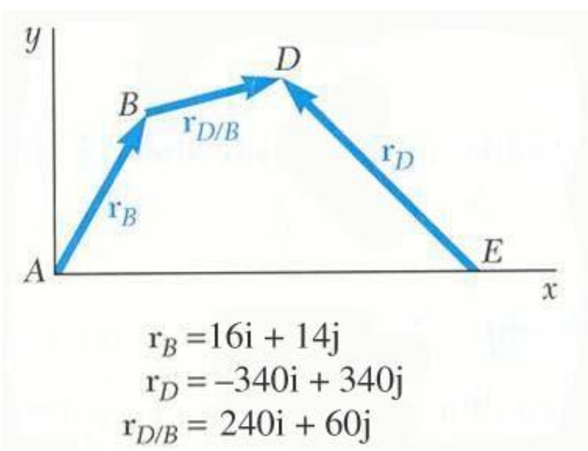
$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

- The angular accelerations are determined by simultaneously solving the component equations for

$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B}$$

# Sample Problem 15.8




SOLUTION:

- The angular velocities are determined by simultaneously solving the component equations for

$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

$$\vec{v}_D = \vec{\omega}_{DE} \times \vec{r}_D = \omega_{DE}\vec{k} \times \left(-340\vec{i} + 340\vec{j}\right)$$

$$= -340\omega_{DE}\vec{i} - 340\omega_{DE}\vec{j}$$

$$\vec{v}_B = \vec{\omega}_{AB} \times \vec{r}_B = 20\vec{k} \times \left(160\vec{i} + 280\vec{j}\right)$$

$$= -5600\vec{i} + 3200\vec{j}$$

$$\vec{v}_{D/B} = \vec{\omega}_{BD} \times \vec{r}_{D/B} = \omega_{BD}\vec{k} \times \left(240\vec{i} + 60\vec{j}\right)$$

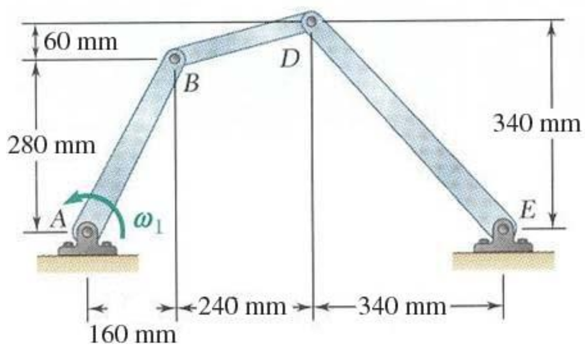
$$= -60\omega_{BD}\vec{i} + 240\omega_{BD}\vec{j}$$

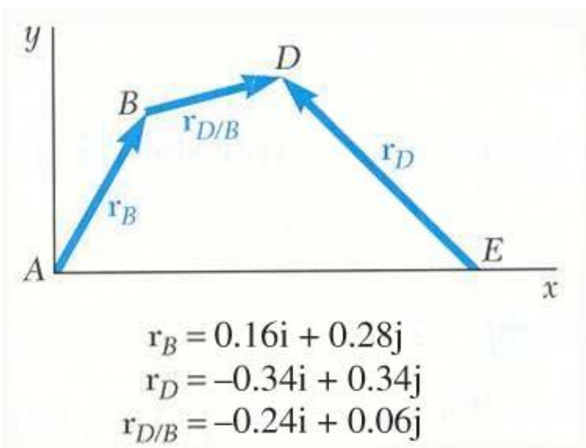
$x$ components: $-17\omega_{DE} = -280 - 3\omega_{BD}$

$y$ components: $-17\omega_{DE} = +160 + 12\omega_{BD}$

$$\vec{\omega}_{BD} = -(29.33\,\text{rad/s})\vec{k} \qquad \vec{\omega}_{DE} = (11.29\,\text{rad/s})\vec{k}$$

## Sample Problem 15.8




(*r* raw expressed in *m*)

- The angular accelerations are determined by simultaneously solving the component equations for

$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B}$$

$$\vec{a}_D = \vec{\alpha}_{DE} \times \vec{r}_D - \omega_{DE}^2 \vec{r}_D$$
$$= \alpha_{DE}\vec{k} \times \left(-0.34\vec{i} + 0.34\vec{j}\right) - (11.29)^2\left(-0.34\vec{i} + 0.34\vec{j}\right)$$
$$= -0.34\alpha_{DE}\vec{i} - 0.34\alpha_{DE}\vec{j} + 43.33\vec{i} - 43.33\vec{j}$$

$$\vec{a}_B = \vec{\alpha}_{AB} \times \vec{r}_B - \omega_{AB}^2 \vec{r}_B = 0 - (20)^2\left(0.16\vec{i} + 0.28\vec{j}\right)$$
$$= -64\vec{i} + 112\vec{j}$$

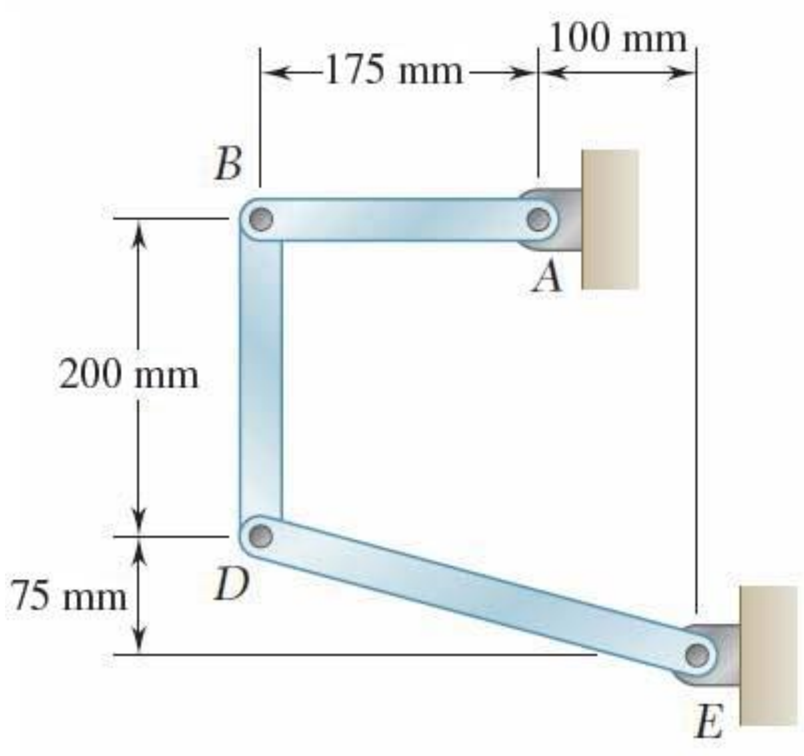
$$\vec{a}_{D/B} = \vec{\alpha}_{BD} \times \vec{r}_{D/B} - \omega_{BD}^2 \vec{r}_{D/B}$$
$$= \alpha_{B/D}\vec{k} \times \left(0.24\vec{i} + 0.06\vec{j}\right) - (29.33)^2\left(0.24\vec{i} + 0.06\vec{j}\right)$$
$$= -0.06\alpha_{B/D}\vec{i} + 0.24\alpha_{B/D}\vec{j} - 206.4\vec{i} - 51.61\vec{j}$$

*x* components: $-0.34\alpha_{DE} + 0.06\alpha_{BD} = -313.7$

*y* components: $-0.34\alpha_{DE} - 0.24\alpha_{BD} = -120.28$

$$\vec{\alpha}_{BD} = -\left(645\,\mathrm{rad/s^2}\right)\vec{k} \qquad \vec{\alpha}_{DE} = \left(809\,\mathrm{rad/s^2}\right)\vec{k}$$

## Group Problem Solving

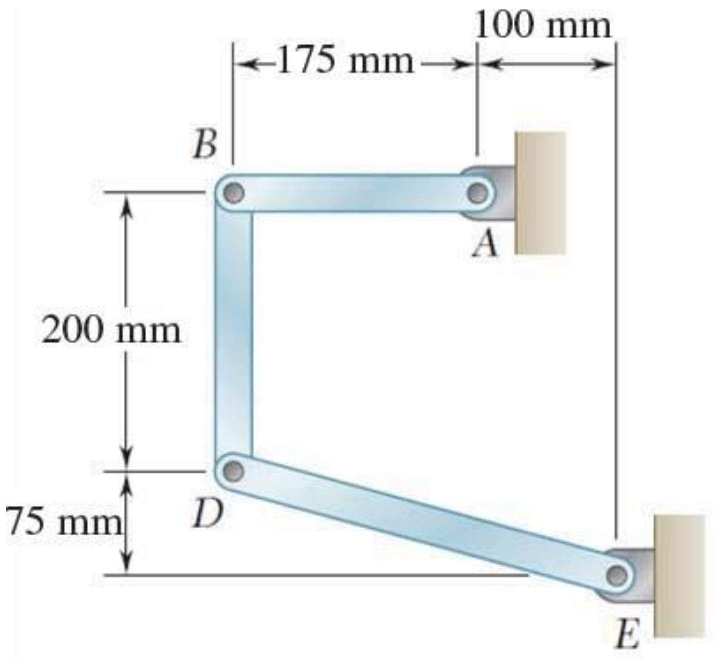



**Knowing that at the instant shown bar *AB* has a constant angular velocity of 4 rad/s clockwise, determine the angular acceleration of bars *BD* and *DE*.**

**Which of the following is true?**

a) The direction of $a_D$ is ↗
b) The angular acceleration of BD must also be constant
c) The direction of the linear acceleration of B is →

## Group Problem Solving




Knowing that at the instant shown bar *AB* has a constant angular velocity of 4 rad/s clockwise, determine the angular acceleration of bars *BD* and *DE*.

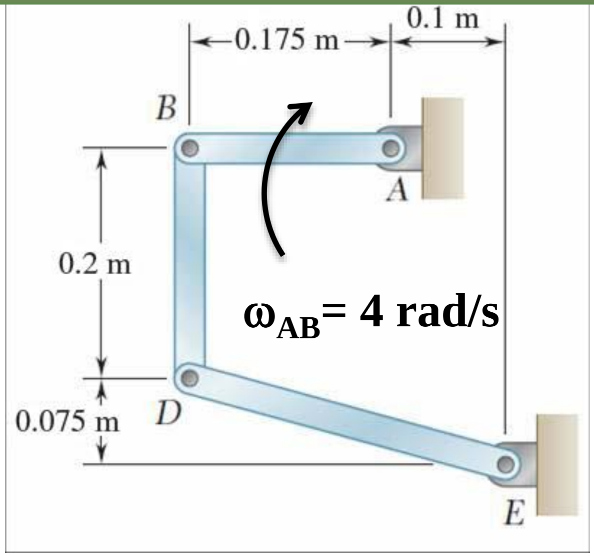
SOLUTION:

- The angular velocities were determined in a previous problem by simultaneously solving the component equations for

$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

- The angular accelerations are now determined by simultaneously solving the component equations for the relative acceleration equation.

## Group Problem Solving




**From our previous problem, we used the relative velocity equations to find that:**

$$\omega_{DE} = 2.55 \text{ rad/s} \quad \omega_{BD} = 0.955 \text{ rad/s}$$

**We can now apply the relative acceleration equation with** $\alpha_{AB} = 0$

**Analyze Bar AB**

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$

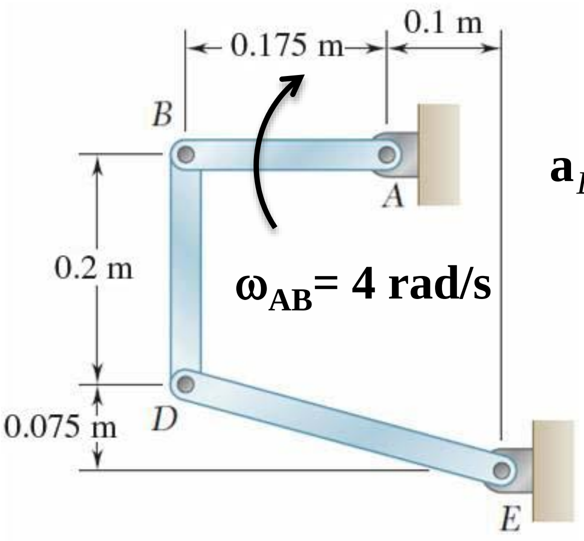
$$\mathbf{a}_B = -\omega_{AB}^2 \mathbf{r}_{B/A} = -(4)^2(-0.175\mathbf{i}) = 2.8 \text{ m/s}^2\mathbf{i}$$

**Analyze Bar BD**

$$\mathbf{a}_D = \mathbf{a}_B + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{D/B} - \omega_{BD}^2 \mathbf{r}_{D/B} = 2.8\mathbf{i} + \alpha_{BD}\mathbf{k} \times (-0.2\mathbf{j}) - (0.95455)^2(-0.2\mathbf{j})$$

$$\mathbf{a}_D = (2.8 + 0.2\,\alpha_{BD})\mathbf{i} + 0.18223\mathbf{j}$$

## Group Problem Solving




### Analyze Bar DE

$$\mathbf{a}_D = \boldsymbol{\alpha}_{DE} \times \mathbf{r}_{D/E} - \omega_{DE}^2 r_{D/E}$$
$$= \alpha_{DE}\mathbf{k} \times (-0.275\mathbf{i} + 0.075\mathbf{j}) - (2.5455)^2(-0.275\mathbf{i} + 0.075\mathbf{j})$$
$$= -0.275\alpha_{DE}\mathbf{j} - 0.075\alpha_{DE}\mathbf{i} + 1.7819\mathbf{i} - 0.486\mathbf{j}$$

$$\mathbf{a}_D = (-0.075\alpha_{DE} + 1.7819)\mathbf{i} - (0.275\alpha_{DE} + 0.486)\mathbf{j}$$

**From previous page, we had:** $\mathbf{a}_D = (2.8 + 0.2\,\alpha_{BD})\mathbf{i} + 0.18223\mathbf{j}$

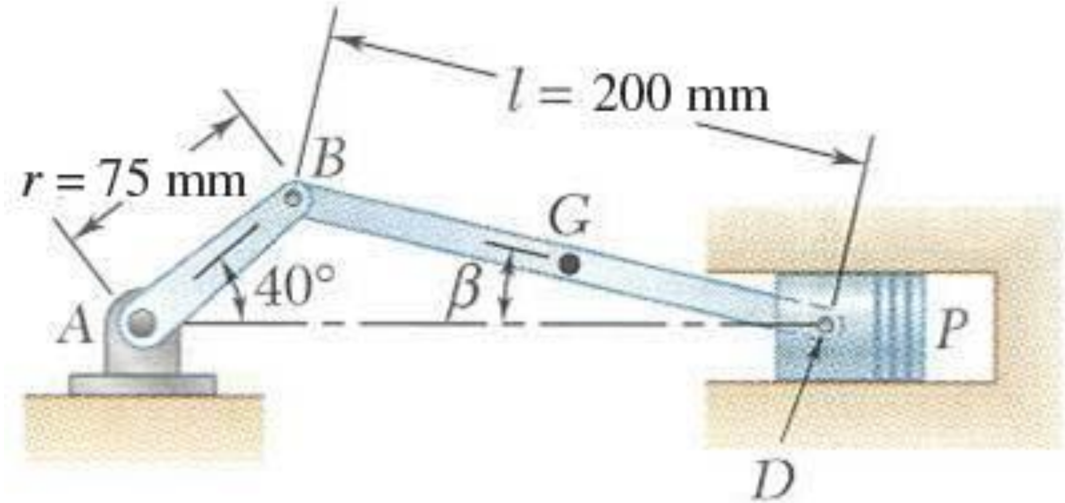
### Equate like components of $\mathbf{a}_D$

j: $0.18223 = -(0.275\alpha_{DE} + 0.486)$

i: $2.8 + 0.2\alpha_{BD} = [-(0.075)(-2.43) + 1.7819]$

$$\alpha_{DE} = -2.43 \text{ rad/s}^2$$
$$\alpha_{BD} = -4.18 \text{ rad/s}^2$$

## Concept Question




**If the clockwise angular velocity of crankshaft AB is constant, which of the following statement is true?**

a) The angular velocity of BD is constant

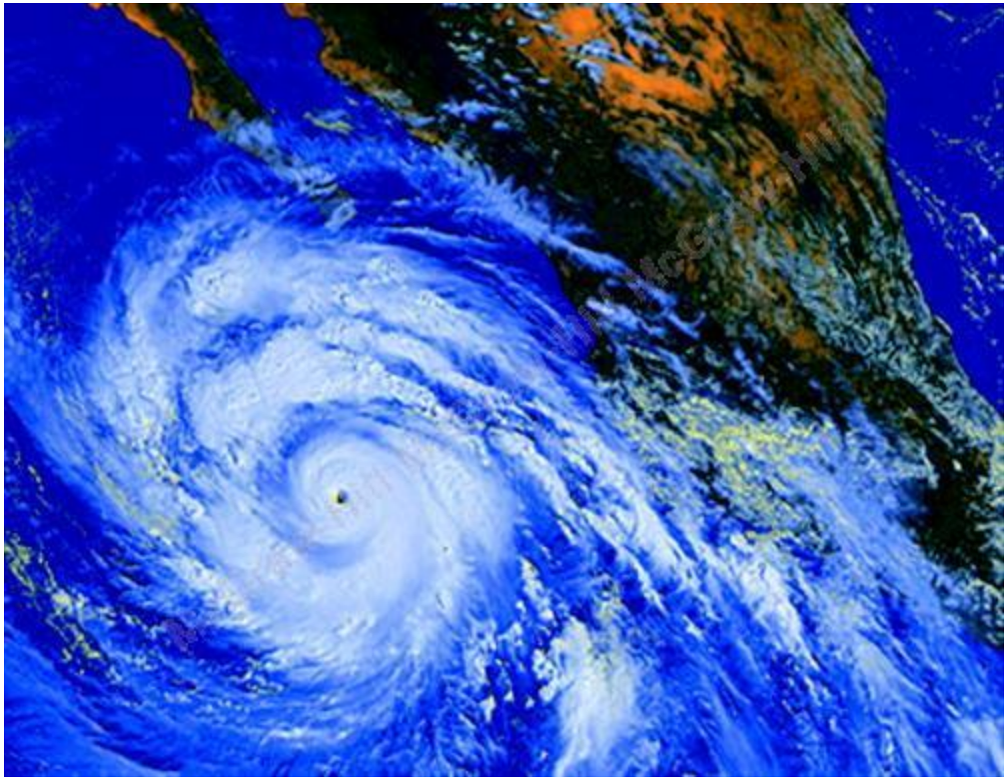
b) The linear acceleration of point B is zero

c) The angular velocity of BD is counterclockwise

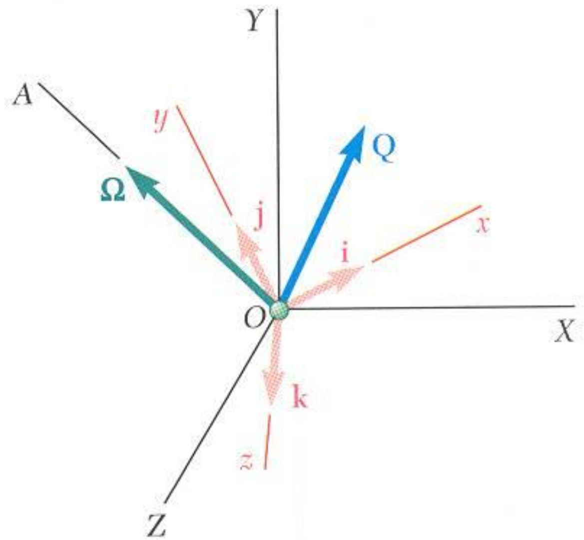
d) The linear acceleration of point B is tangent to the path

# Applications

**Rotating coordinate systems are often used to analyze mechanisms (such as amusement park rides) as well as weather patterns.**



# Rate of Change With Respect to a Rotating Frame




- Frame *OXYZ* is fixed.
- Frame *Oxyz* rotates about fixed axis *OA* with angular velocity $\vec{\Omega}$
- Vector function $\vec{Q}(t)$ varies in direction and magnitude.

- With respect to the rotating *Oxyz* frame,

$$\vec{Q} = Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}$$

$$\left(\dot{\vec{Q}}\right)_{Oxyz} = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k}$$

- With respect to the fixed *OXYZ* frame,

$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k} + Q_x\dot{\vec{i}} + Q_y\dot{\vec{j}} + Q_z\dot{\vec{k}}$$

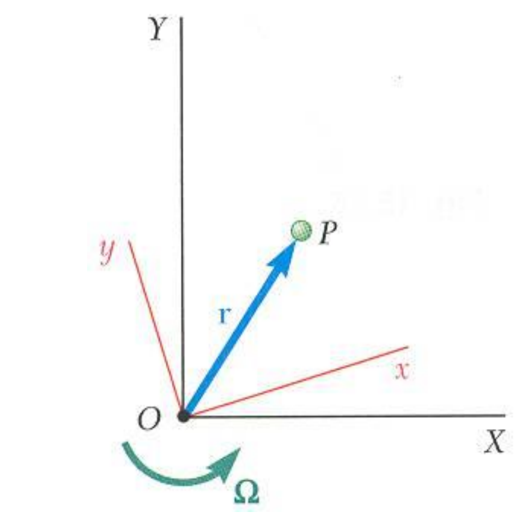
- $\dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k} = \left(\dot{\vec{Q}}\right)_{Oxyz}$ = rate of change with respect to rotating frame.

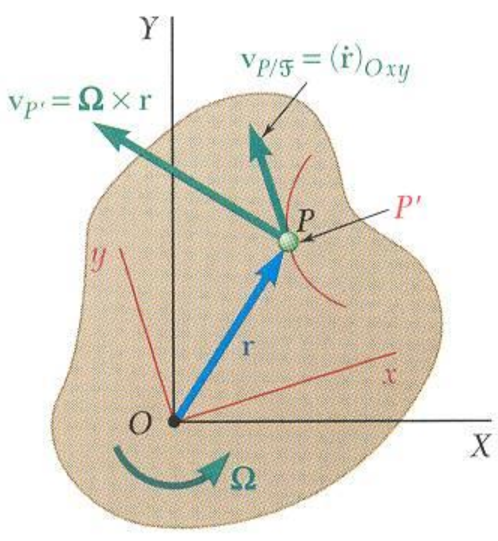
- If $\vec{Q}$ were fixed within *Oxyz* then $\left(\dot{\vec{Q}}\right)_{OXYZ}$ is equivalent to velocity of a point in a rigid body attached to *Oxyz* and $Q_x\dot{\vec{i}} + Q_y\dot{\vec{j}} + Q_z\dot{\vec{k}} = \vec{\Omega}\times\vec{Q}$

- With respect to the fixed *OXYZ* frame,

$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \left(\dot{\vec{Q}}\right)_{Oxyz} + \vec{\Omega}\times\vec{Q}$$

# Coriolis Acceleration




- Frame *OXY* is fixed and frame *Oxy* rotates with angular velocity $\vec{\Omega}$.
- Position vector $\vec{r}_P$ for the particle *P* is the same in both frames but the rate of change depends on the choice of frame.
- The absolute velocity of the particle *P* is

$$\vec{v}_P = \left(\dot{\vec{r}}\right)_{OXY} = \vec{\Omega} \times \vec{r} + \left(\dot{r}\right)_{Oxy}$$

- Imagine a rigid slab attached to the rotating frame *Oxy* or $\mathfrak{F}$ for short. Let *P* ′ be a point on the slab which corresponds instantaneously to position of particle *P*.

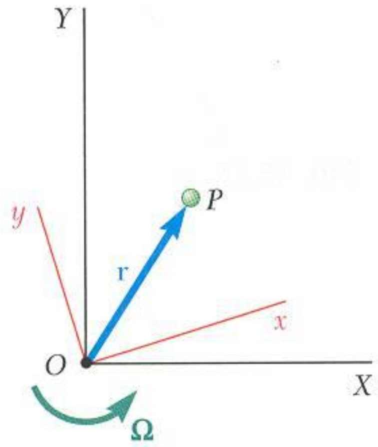
$\vec{v}_{P/\mathfrak{F}} = \left(\dot{\vec{r}}\right)_{Oxy}$ = velocity of *P* along its path on the slab

$\vec{v}_{P'}$ = absolute velocity of point *P* ′ on the slab

- Absolute velocity for the particle P may be written as

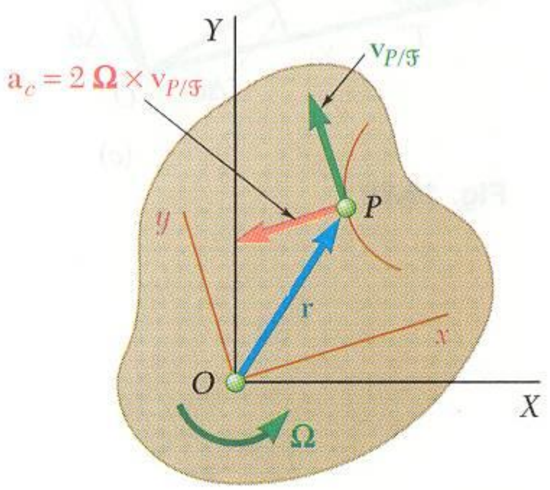
$$\vec{v}_P = \vec{v}_{P¢} + \vec{v}_{P/\mathbf{F}}$$

# Coriolis Acceleration




$$\vec{v}_P = \vec{\Omega}\times\vec{r} + (\dot{r})_{Oxy}$$

$$= \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$




- Absolute acceleration for the particle $P$ is

$$\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\dot{\vec{r}})_{OXY} + \frac{d}{dt}\left[(\dot{\vec{r}})_{Oxy}\right]$$

but, $$(\dot{\vec{r}})_{OXY} = \vec{\Omega}\times\vec{r} + (\dot{\vec{r}})_{Oxy}$$

$$\frac{d}{dt}\left[(\dot{\vec{r}})_{Oxy}\right] = (\ddot{\vec{r}})_{Oxy} + \vec{\Omega}\times(\dot{\vec{r}})_{Oxy}$$

$$\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy} + (\ddot{\vec{r}})_{Oxy}$$

- Utilizing the conceptual point $P'$ on the slab,

$$\vec{a}_{P'} = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\vec{a}_{P/\mathfrak{F}} = (\ddot{\vec{r}})_{Oxy}$$

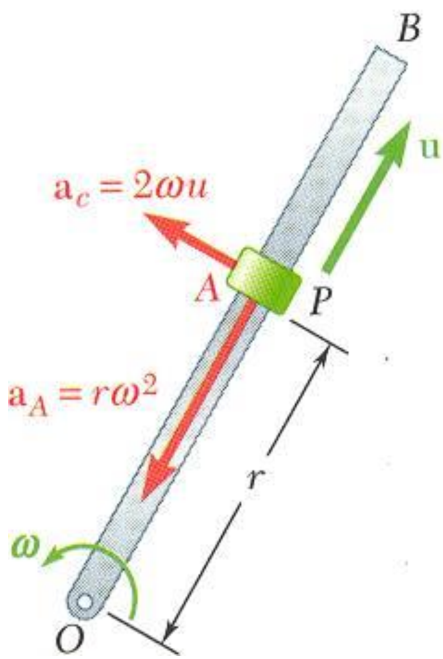
- Absolute acceleration for the particle $P$ becomes

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathfrak{F}} + 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy}$$

$$= \vec{a}_{P'} + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$$

$$\vec{a}_c = 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy} = 2\vec{\Omega}\times\vec{v}_{P/\mathfrak{F}} = \text{Coriolis acceleration}$$

# Coriolis Acceleration




- Consider a collar $P$ which is made to slide at constant relative velocity $u$ along rod $OB$. The rod is rotating at a constant angular velocity $\omega$. The point $A$ on the rod corresponds to the instantaneous position of $P$.
- Absolute acceleration of the collar is

$$\vec{a}_P = \vec{a}_A + \vec{a}_{P/\mathbf{F}} + \vec{a}_c$$

where

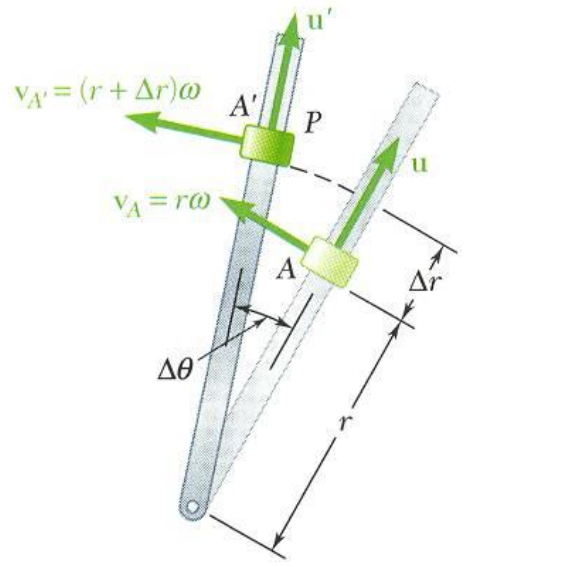
$$\vec{a}_A = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) \qquad a_A = r\omega^2$$

$$\vec{a}_{P/\mathscr{F}} = \left(\ddot{\vec{r}}\right)_{Oxy} = 0$$

$$\vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathscr{F}} \qquad a_c = 2\omega u$$

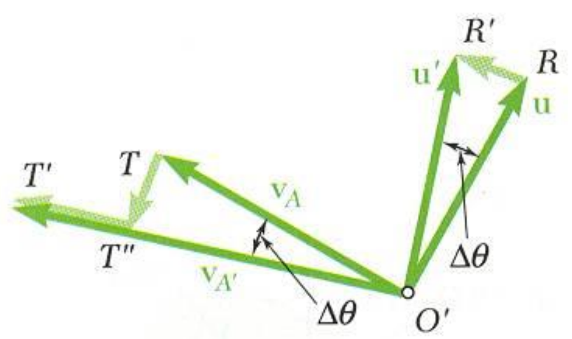
- The absolute acceleration consists of the radial and tangential vectors shown

# Coriolis Acceleration




at $t$, $\qquad \vec{v} = \vec{v}_A + \vec{u}$

at $t + \Delta t$, $\qquad \vec{v}\,' = \vec{v}_{A'} + \vec{u}\,'$




- Change in velocity over $\Delta t$ is represented by the sum of three vectors

$$\Delta\vec{v} = \overrightarrow{RR'} + \overrightarrow{TT''} + \overrightarrow{T''T'}$$

- $\overrightarrow{TT''}$ is due to change in direction of the velocity of point $A$ on the rod,

$$\lim_{\Delta t \to 0} \frac{\overrightarrow{TT''}}{\Delta t} = \lim_{\Delta t \to 0} v_A \frac{\Delta\theta}{\Delta t} = r\omega\omega = r\omega^2 = a_A$$

$$\text{recall,} \quad \vec{a}_A = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) \qquad a_A = r\omega^2$$

- $\overrightarrow{RR'}$ and $\overrightarrow{T''T'}$ result from combined effects of relative motion of $P$ and rotation of the rod

$$\lim_{\Delta t \to 0}\left(\frac{\overrightarrow{RR'}}{\Delta t} + \frac{\overrightarrow{T''T'}}{\Delta t}\right) = \lim_{\Delta t \to 0}\left(u\frac{\Delta\theta}{\Delta t} + \omega\frac{\Delta r}{\Delta t}\right)$$

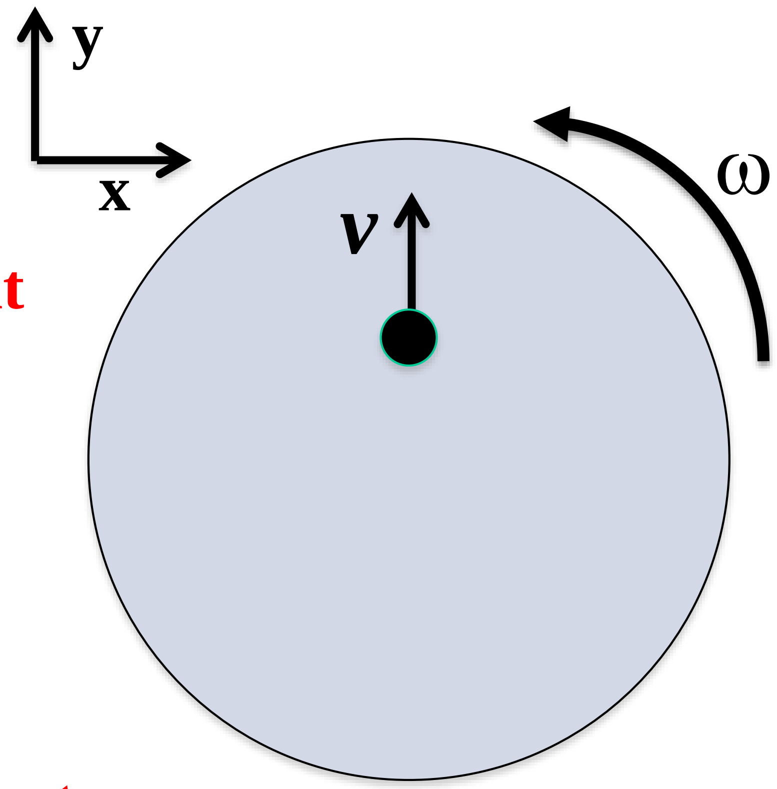
$$= u\omega + \omega u = 2\omega u$$

$$\text{recall,} \quad \vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathfrak{F}} \qquad a_c = 2\omega u$$

## Concept Question

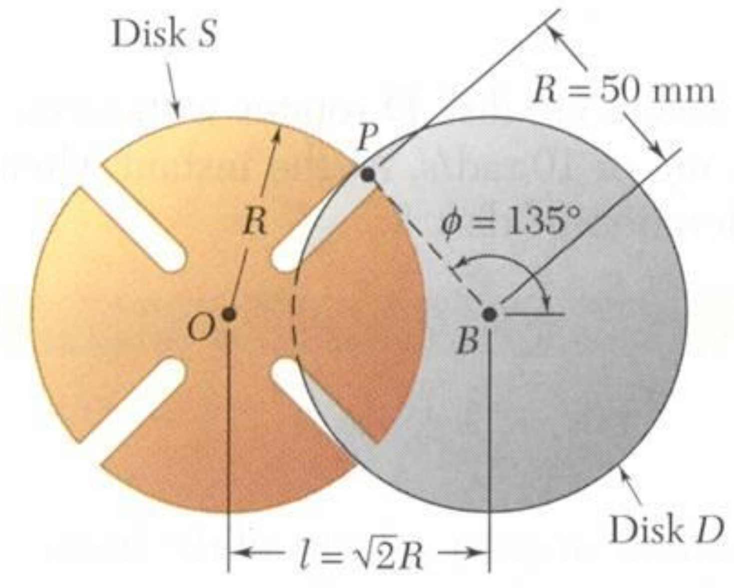
**You are walking with a constant velocity with respect to the platform, which rotates with a constant angular velocity w. At the instant shown, in which direction(s) will you experience an acceleration (choose all that apply)?**

a) +x
b) -x
c) +y
d) -y
e) Acceleration = 0




$$\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) + 2\vec{\Omega}\times\left(\dot{\vec{r}}\right)_{Oxy} + \left(\ddot{\vec{r}}\right)_{Oxy}$$

# Sample Problem 15.9




Disk D of the Geneva mechanism rotates with constant counterclockwise angular velocity $\omega_D$ = 10 rad/s.

At the instant when $\phi = 150^{\circ}$, determine (*a*) the angular velocity of disk *S*, and (*b*) the velocity of pin *P* relative to disk *S*.

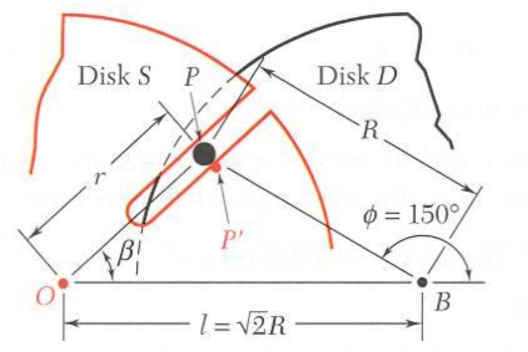
SOLUTION:

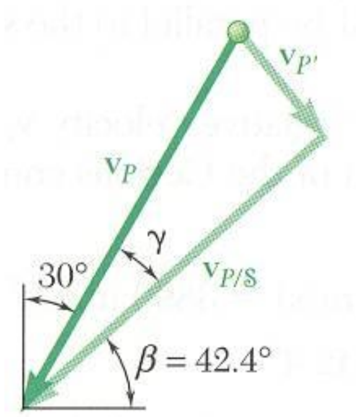
- The absolute velocity of the point *P* may be written as

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/s}$$

- Magnitude and direction of velocity $\vec{v}_P$ of pin *P* are calculated from the radius and angular velocity of disk *D*.
- Direction of velocity $\vec{v}_{P'}$ of point *P*' on *S* coinciding with *P* is perpendicular to radius *OP*.
- Direction of velocity $\vec{v}_{P/s}$ of *P* with respect to *S* is parallel to the slot.
- Solve the vector triangle for the angular velocity of *S* and relative velocity of *P*.

## Sample Problem 15.9




SOLUTION:

- The absolute velocity of the point *P* may be written as

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/s}$$

- Magnitude and direction of absolute velocity of pin *P* are calculated from radius and angular velocity of disk *D*.

$$v_P = R\omega_D = (50\text{ mm})(10\text{ rad/s}) = 500\text{ mm/s}$$

- Direction of velocity of *P* with respect to *S* is parallel to slot. From the law of cosines,

$$r^2 = R^2 + l^2 - 2Rl\cos 30° = 0.551R^2 \qquad r = 37.1\text{ mm}$$

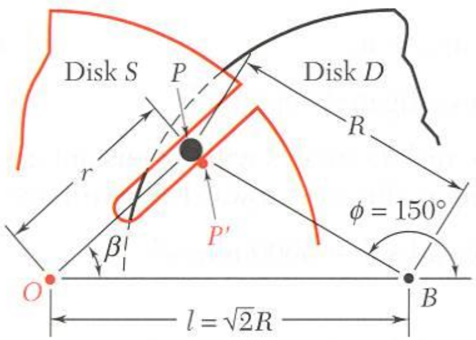
From the law of cosines,

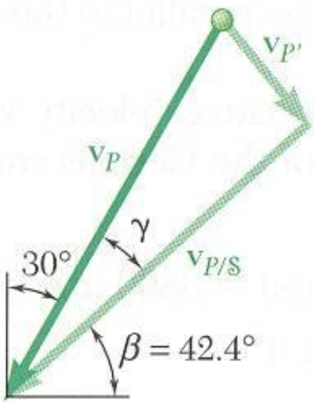
$$\frac{\sin\beta}{\text{R}} = \frac{\sin 30°}{r} \qquad \sin\beta = \frac{\sin 30°}{0.742} \qquad \beta = 42.4°$$

The interior angle of the vector triangle is

$$\gamma = 90° - 42.4° - 30° = 17.6°$$

# Sample Problem 15.9




$v_P = 500 \text{ mm/s}$

- Direction of velocity of point *P*' on *S* coinciding with *P* is perpendicular to radius *OP*. From the velocity triangle,

$$v_{P'} = v_P \sin\gamma = (500 \text{ mm/s})\sin 17.6° = 151.2 \text{ mm/s}$$

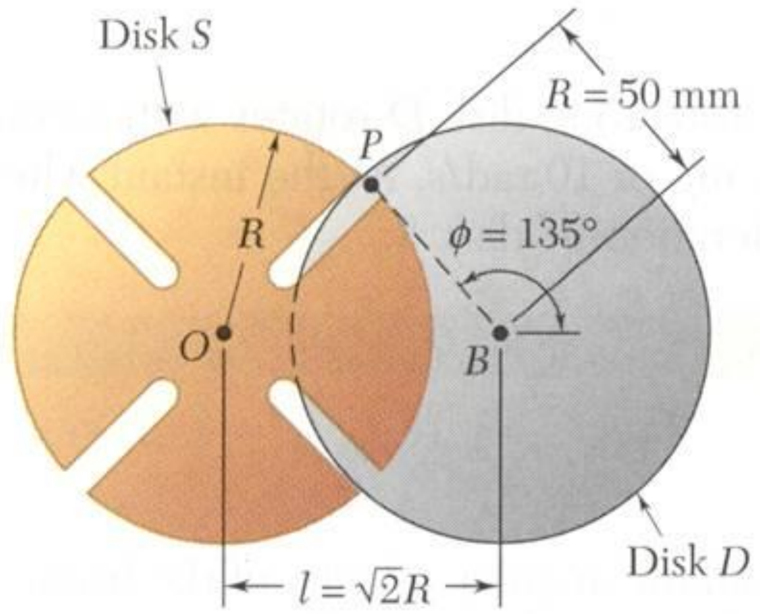
$$= r\omega_s \qquad \omega_s = \frac{151.2 \text{ mm/s}}{37.1 \text{ mm}}$$

$$\vec{\omega}_s = (-4.08 \text{ rad/s})\vec{k}$$

$$v_{P/s} = v_P \cos\gamma = (500 \text{ m/s})\cos 17.6°$$

$$\vec{v}_{P/s} = (477 \text{ m/s})(-\cos 42.4°\vec{i} - \sin 42.4°\vec{j})$$

# Sample Problem 15.10




In the Geneva mechanism, disk *D* rotates with a constant counter-clockwise angular velocity of 10 rad/s. At the instant when φ = 150°, determine angular acceleration of disk *S*.

SOLUTION:

- The absolute acceleration of the pin *P* may be expressed as

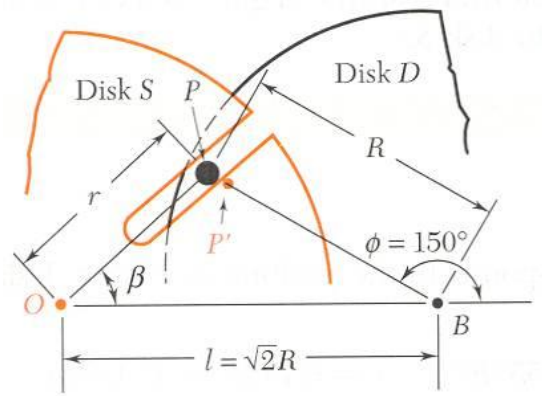
$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/s} + \vec{a}_c$$

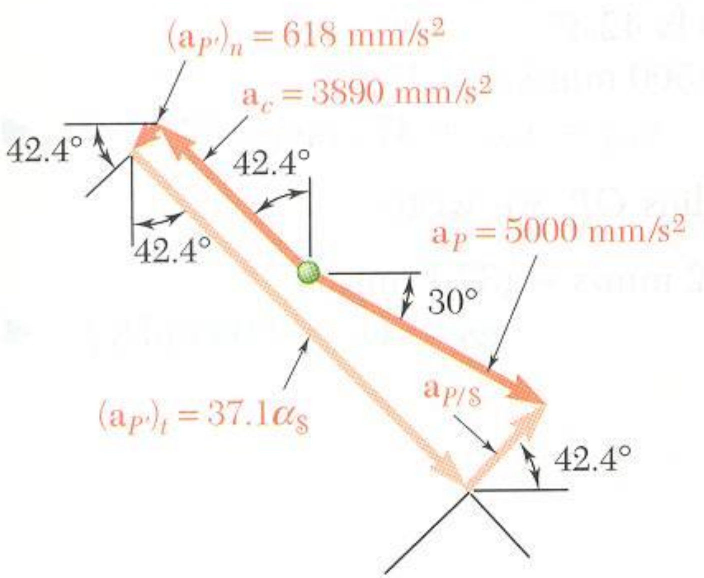
- The instantaneous angular velocity of Disk *S* is determined as in Sample Problem 15.9.

- The only unknown involved in the acceleration equation is the instantaneous angular acceleration of Disk *S*.

- Resolve each acceleration term into the component parallel to the slot. Solve for the angular acceleration of Disk *S*.

# Sample Problem 15.10




SOLUTION:

- Absolute acceleration of the pin $P$ may be expressed as

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/s} + \vec{a}_c$$

- From Sample Problem 15.9.

$$\beta = 42.4^\circ \qquad \vec{\omega}_S = (-4.08\,\mathrm{rad/s})\vec{k}$$

$$\vec{v}_{P/s} = (477\,\mathrm{mm/s})(-\cos 42.4^\circ \vec{i} - \sin 42.4^\circ \vec{j})$$

- Considering each term in the acceleration equation,

$$a_P = R\omega_D^2 = (500\,\mathrm{mm})(10\,\mathrm{rad/s})^2 = 5000\,\mathrm{mm/s^2}$$

$$\vec{a}_P = (5000\,\mathrm{mm/s^2})(\cos 30^\circ \vec{i} - \sin 30^\circ \vec{j})$$

$$\vec{a}_{P'} = (\vec{a}_{P'})_n + (\vec{a}_{P'})_t$$

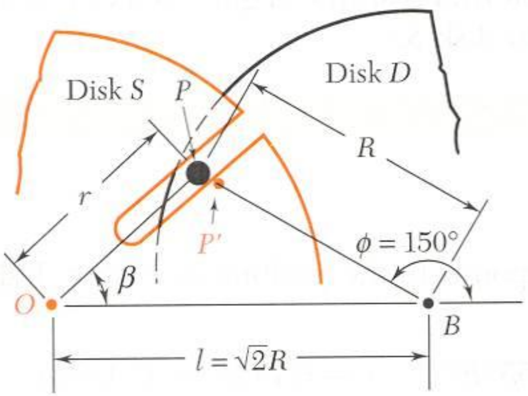
$$(\vec{a}_{P'})_n = (r\omega_S^2)(-\cos 42.4^\circ \vec{i} - \sin 42.4^\circ \vec{j})$$

$$(\vec{a}_{P'})_t = (r\alpha_S)(-\sin 42.4^\circ \vec{i} + \cos 42.4^\circ \vec{j})$$

$$(\vec{a}_{P'})_t = (\alpha_S)(37.1\,\mathrm{mm})(-\sin 42.4^\circ \vec{i} + \cos 42.4^\circ \vec{j})$$

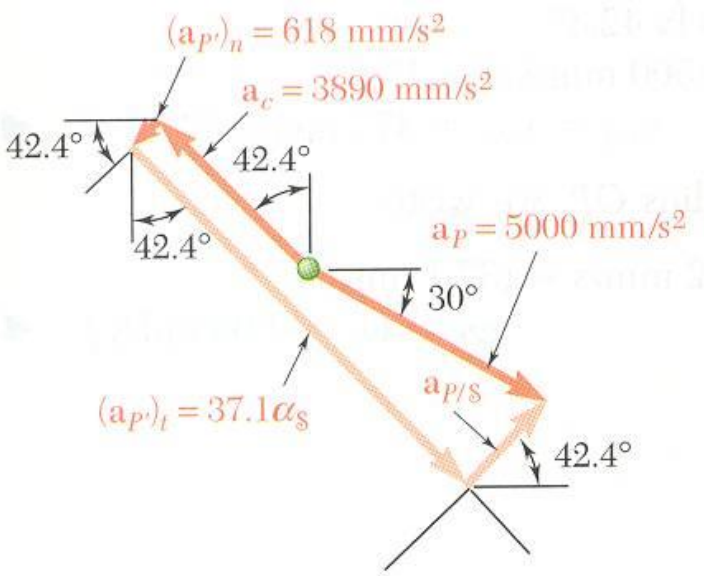
note: $\alpha_S$ may be positive or negative

## Sample Problem 15.10




- The direction of the Coriolis acceleration is obtained by rotating the direction of the relative velocity $\vec{v}_{P/s}$ by 90° in the sense of $\omega_S$.

$$\vec{a}_c = (2\omega_S v_{P/s})(-\sin 42.4°\vec{i} + \cos 42.4\vec{j})$$
$$= 2(4.08\,\text{rad/s})(477\,\text{mm/s})(-\sin 42.4°\vec{i} + \cos 42.4\vec{j})$$
$$= (3890\,\text{mm/s}^2)(-\sin 42.4°\vec{i} + \cos 42.4\vec{j})$$




- The relative acceleration $\vec{a}_{P/s}$ must be parallel to the slot.

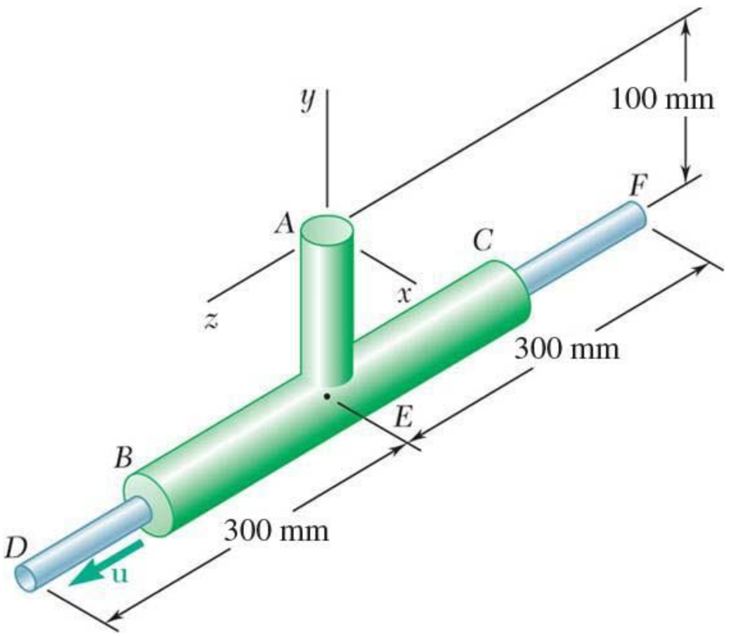
- Equating components of the acceleration terms perpendicular to the slot,

$$37.1\alpha_S + 3890 - 5000\cos 17.7° = 0$$
$$\alpha_S = -233\,\text{rad/s}$$

$$\vec{\alpha}_S = (-233\,\text{rad/s})\vec{k}$$

## Group Problem Solving

The sleeve *BC* is welded to an arm that rotates about stationary point *A* with a constant angular velocity **ω** = (3 rad/s) **j**. In the position shown rod *DF* is being moved to the left at a constant speed *u* = 400 mm/s relative to the sleeve. Determine the acceleration of Point *D*.




SOLUTION:

- The absolute acceleration of point *D* may be expressed as

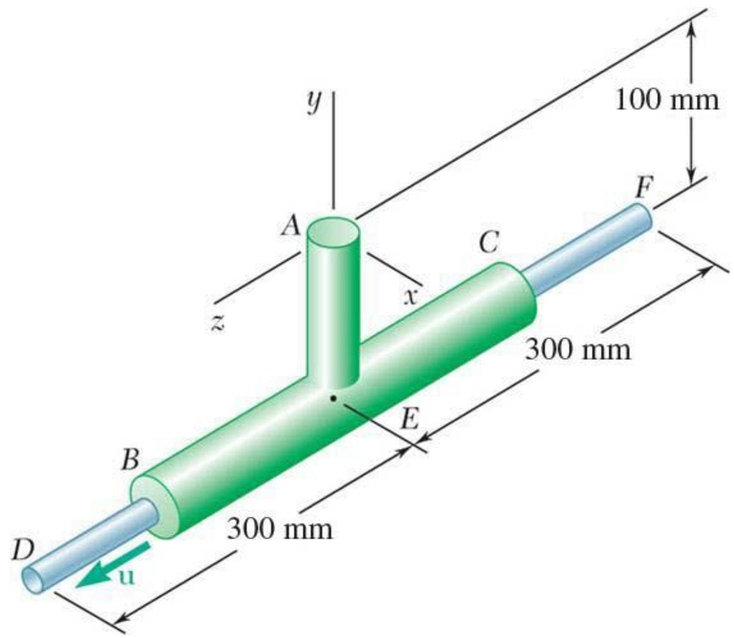
$$\vec{a}_D = \vec{a}_{D'} + \vec{a}_{D/BC} + \vec{a}_c$$

- Determine the acceleration of the virtual point D'.
- Calculate the Coriolis acceleration.
- Add the different components to get the overall acceleration of point D.

## Group Problem Solving

Given: u= 400 mm/s, **ω** = (3 rad/s) **j**.

Find: $\mathbf{a}_D$




**Write overall expression for $\mathbf{a}_D$**

$$\vec{a}_D = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) + 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxy} + \left(\ddot{\vec{r}}\right)_{Oxy}$$

**Do any of the terms go to zero?**

$$\vec{a}_D = \cancel{\dot{\vec{\Omega}} \times \vec{r}} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) + 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxy} + \cancel{\left(\ddot{\vec{r}}\right)_{Oxy}}$$

**Determine the normal acceleration term of the virtual point D'**

$$\mathbf{a}_{D'} = \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$$ where r is from A to D

$$= (3 \text{ rad/s})\mathbf{j} \times \left\{(3 \text{ rad/s})\mathbf{j} \times [-(100 \text{ mm})\mathbf{j} + (300 \text{ mm})\mathbf{k}]\right\}$$

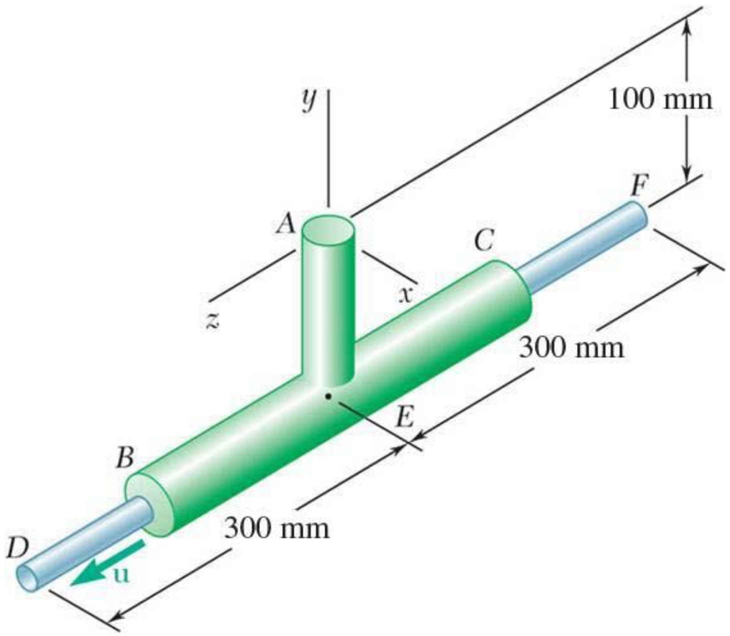
$$= -(2700 \text{ mm/s}^2)\mathbf{k}$$

# Group Problem Solving

$$\vec{a}_D = \dot{\vec{W}} \times \vec{r} + \vec{W} \times \left(\vec{W} \times \vec{r}\right) + 2\vec{W} \times \left(\dot{\vec{r}}\right)_{Oxy} + \left(\ddot{\vec{r}}\right)_{Oxy}$$

**Determine the Coriolis acceleration of point**

$$\mathbf{a}_C = 2\boldsymbol{\omega} \times \mathbf{v}_{D/F}$$
$$= 2(3 \text{ rad/s})\mathbf{j} \times (400 \text{ mm/s})\mathbf{k}$$
$$= (2400 \text{ mm/s}^2)\mathbf{i}$$



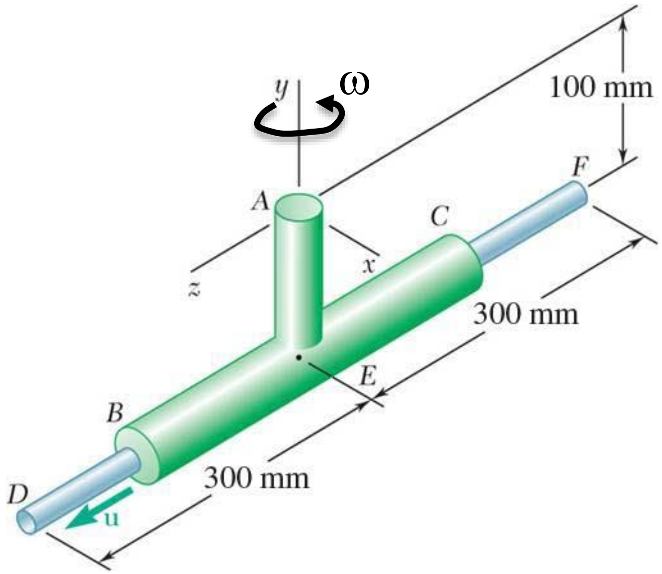

**Add the different components to obtain the total acceleration of point *D***

$$\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/F} + \mathbf{a}_C$$
$$= -(2700 \text{ mm/s}^2)\mathbf{k} + 0 + (2400 \text{ mm/s}^2)\mathbf{i}$$

$$\boxed{\mathbf{a}_D = (2400 \text{ mm/s}^2)\mathbf{i} - (2700 \text{ mm/s}^2)\mathbf{k}}$$

## Group Problem Solving

**In the previous problem, *u* and *ω* were both constant.**



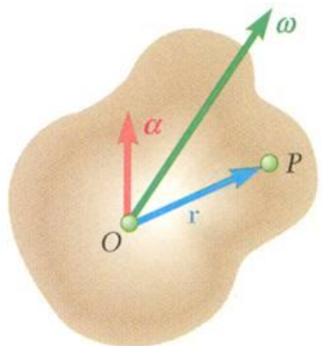

**What would happen if *u* was increasing?**

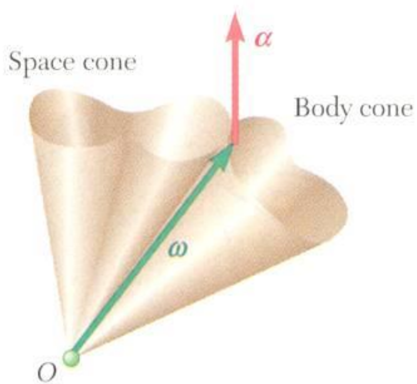
a) The x-component of $a_D$ would increase
b) The y-component of $a_D$ would increase
c) The z-component of $a_D$ would increase
d) The acceleration of $a_D$ would stay the same

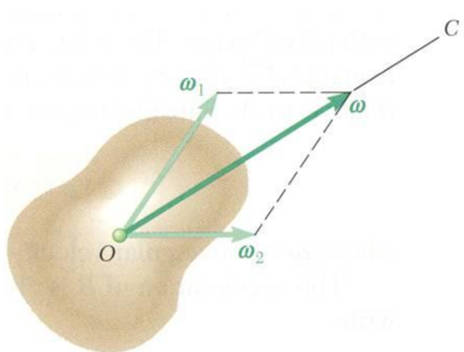
**What would happen if *ω* was increasing?**

a) The x-component of $a_D$ would increase
b) The y-component of $a_D$ would increase
c) The z-component of $a_D$ would increase
d) The acceleration of $a_D$ would stay the same

# Motion About a Fixed Point




- The most general displacement of a rigid body with a fixed point *O* is equivalent to a rotation of the body about an axis through *O*.
- With the instantaneous axis of rotation and angular velocity $\vec{\omega}$, the velocity of a particle *P* of the body is

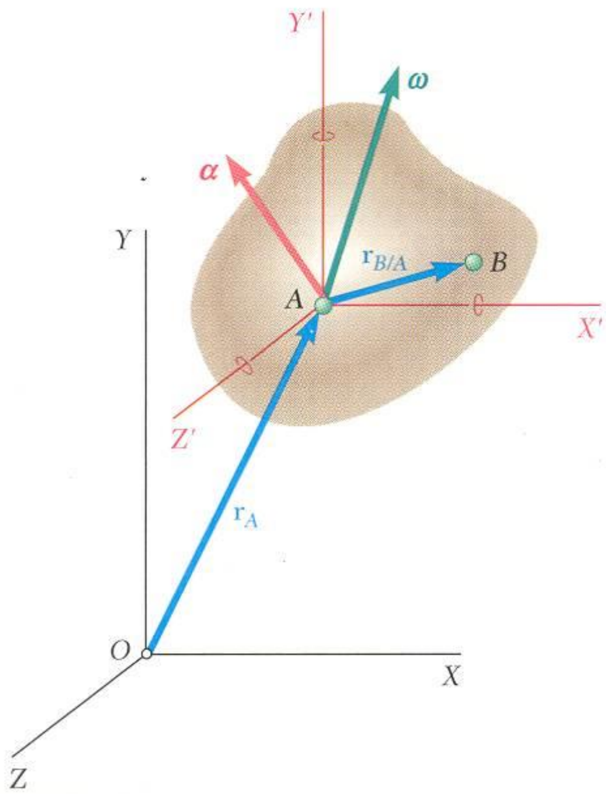
$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

and the acceleration of the particle *P* is

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \qquad \vec{\alpha} = \frac{d\vec{\omega}}{dt}.$$

- The angular acceleration $\vec{\alpha}$ represents the velocity of the tip of $\vec{\omega}$.
- As the vector $\vec{\omega}$ moves within the body and in space, it generates a body cone and space cone which are tangent along the instantaneous axis of rotation.
- Angular velocities have magnitude and direction and obey parallelogram law of addition. They are vectors.

# General Motion




- For particles $A$ and $B$ of a rigid body,

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

- Particle $A$ is fixed within the body and motion of the body relative to $AX'Y'Z'$ is the motion of a body with a fixed point

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

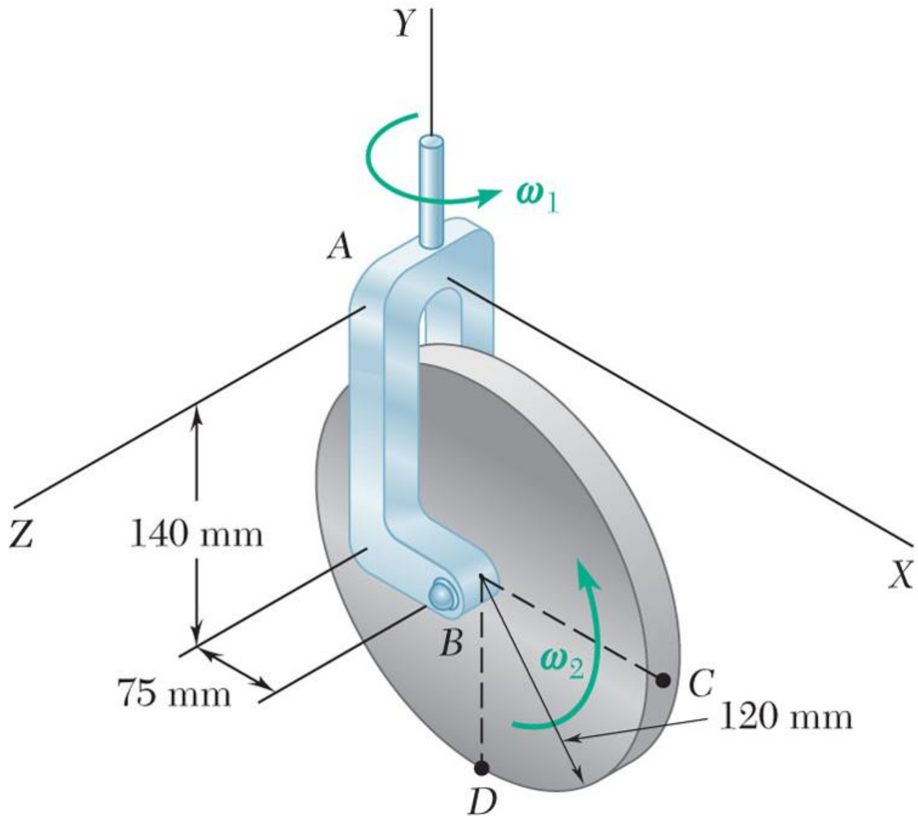
- Similarly, the acceleration of the particle $P$ is

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$
$$= \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right)$$

- Most general motion of a rigid body is equivalent to:
  - a translation in which all particles have the same velocity and acceleration of a reference particle $A$, and
  - of a motion in which particle $A$ is assumed fixed.

## Concept Question

**The figure depicts a model of a coaster wheel. If both $\omega_1$ and $\omega_2$ are constant, what is true about the angular acceleration of the wheel?**




a) It is zero.

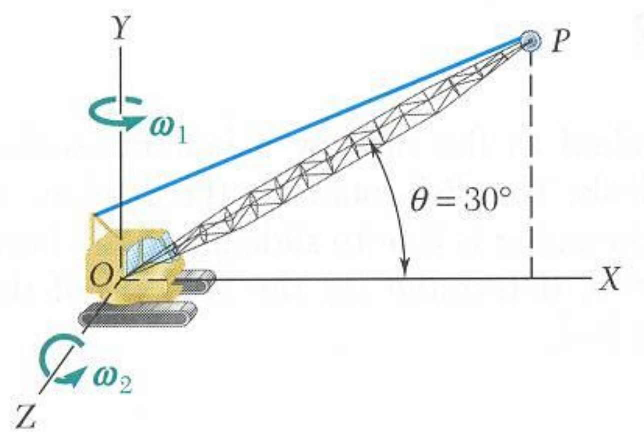
b) It is in the +x direction

c) It is in the +z direction

d) It is in the -x direction

e) It is in the -z direction

## Sample Problem 15.11




The crane rotates with a constant angular velocity $\omega_1$ = 0.30 rad/s and the boom is being raised with a constant angular velocity $\omega_2$ = 0.50 rad/s. The length of the boom is $l$ = 12 m.

Determine:

- angular velocity of the boom,
- angular acceleration of the boom,
- velocity of the boom tip, and
- acceleration of the boom tip.

SOLUTION:

With $\vec{\omega}_1 = 0.30\vec{j} \quad \vec{\omega}_2 = 0.50\vec{k}$

$$\vec{r} = 12\left(\cos 30°\vec{i} + \sin 30°\vec{j}\right)$$
$$= 10.39\vec{i} + 6\vec{j}$$

- Angular velocity of the boom,

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

- Angular acceleration of the boom,

$$\vec{\alpha} = \dot{\vec{\omega}}_1 + \dot{\vec{\omega}}_2 = \dot{\vec{\omega}}_2 = \left(\dot{\vec{\omega}}_2\right)_{Oxyz} + \vec{\Omega} \times \vec{\omega}_2$$
$$= \vec{\omega}_1 \times \vec{\omega}_2$$

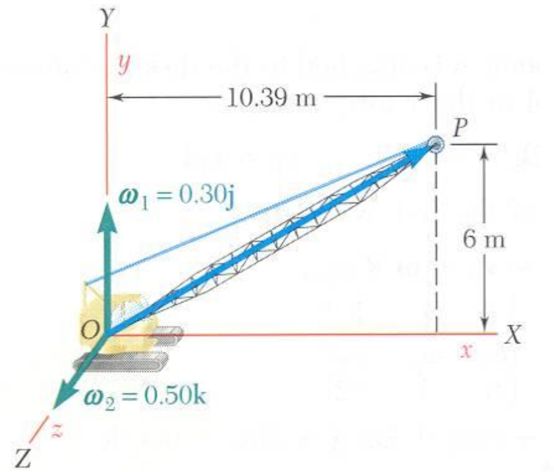
- Velocity of boom tip,

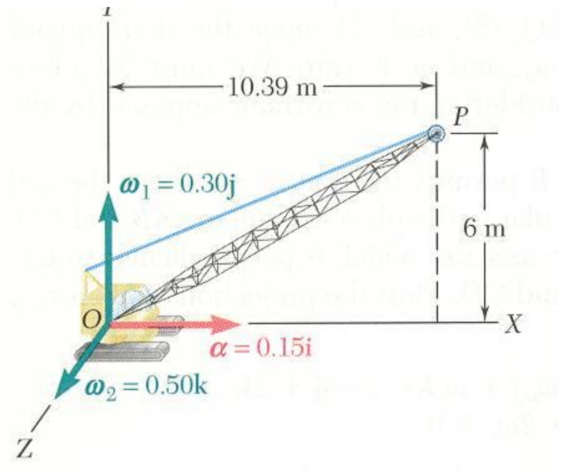
$$\vec{v} = \vec{\omega} \times \vec{r}$$

- Acceleration of boom tip,

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

# Sample Problem 15.11




$$\vec{\omega}_1 = 0.30\vec{j} \quad \vec{\omega}_2 = 0.50\vec{k}$$

$$\vec{r} = 10.39\vec{i} + 6\vec{j}$$

SOLUTION:

- Angular velocity of the boom,

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

$$\boxed{\vec{\omega} = (0.30\,\mathrm{rad/s})\vec{j} + (0.50\,\mathrm{rad/s})\vec{k}}$$

- Angular acceleration of the boom,

$$\vec{\alpha} = \dot{\vec{\omega}}_1 + \dot{\vec{\omega}}_2 = \dot{\vec{\omega}}_2 = \left(\dot{\vec{\omega}}_2\right)_{Oxyz} + \vec{\Omega} \times \vec{\omega}_2$$

$$= \vec{\omega}_1 \times \vec{\omega}_2 = (0.30\,\mathrm{rad/s})\vec{j} \times (0.50\,\mathrm{rad/s})\vec{k}$$

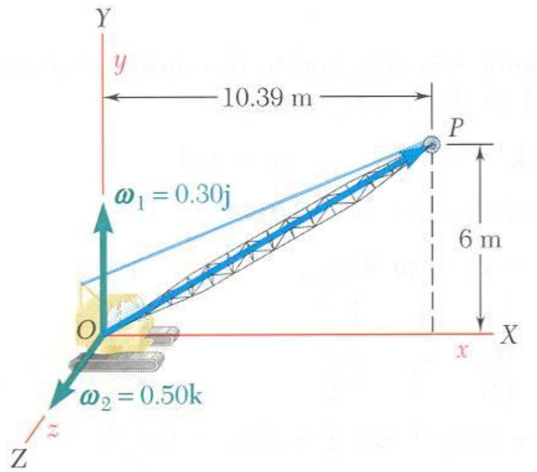
$$\boxed{\vec{\alpha} = \left(0.15\,\mathrm{rad/s^2}\right)\vec{i}}$$

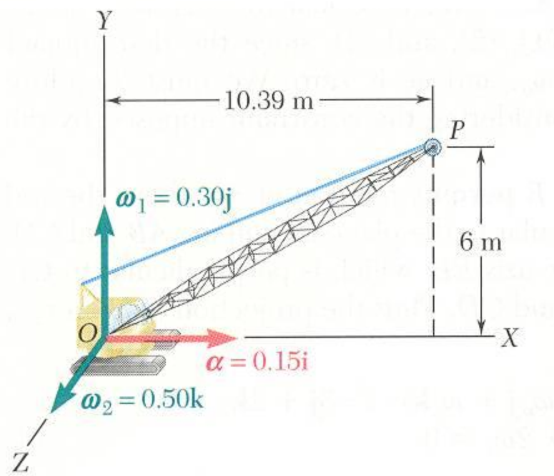
- Velocity of boom tip,

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0.3 & 0.5 \\ 10.39 & 6 & 0 \end{vmatrix}$$

$$\boxed{\vec{v} = -(3.54\,\mathrm{m/s})\vec{i} + (5.20\,\mathrm{m/s})\vec{j} - (3.12\,\mathrm{m/s})\vec{k}}$$

# Sample Problem 15.11




$$\vec{\omega}_1 = 0.30\vec{j} \qquad \vec{\omega}_2 = 0.50\vec{k}$$

$$\vec{r} = 10.39\vec{i} + 6\vec{j}$$

- Acceleration of boom tip,

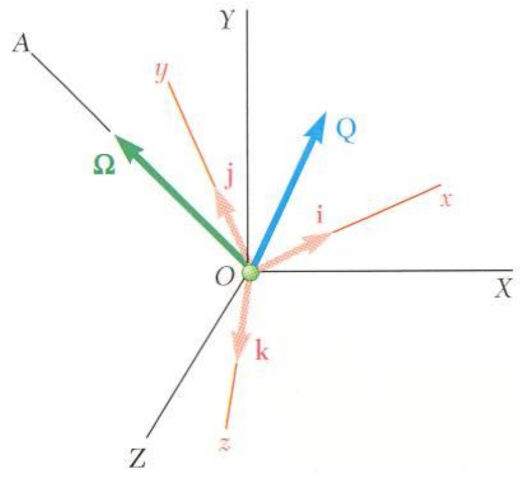
$$\vec{a} = \vec{\alpha}\times\vec{r} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) = \vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{v}$$

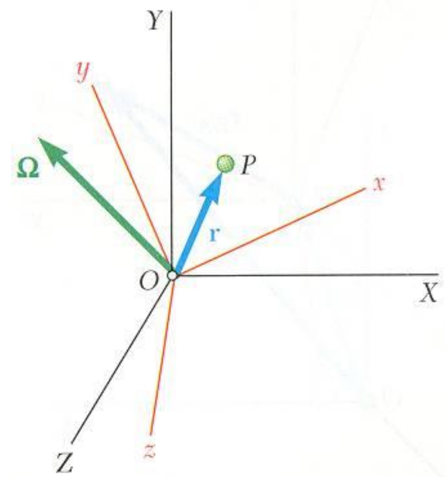
$$\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.15 & 0 & 0 \\ 10.39 & 6 & 0 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0.30 & 0.50 \\ -3 & 5.20 & -3.12 \end{vmatrix}$$

$$= 0.90\vec{k} - 0.94\vec{i} - 2.60\vec{i} - 1.50\vec{j} + 0.90\vec{k}$$

$$\vec{a} = -\left(3.54\,\mathrm{m/s^2}\right)\vec{i} - \left(1.50\,\mathrm{m/s^2}\right)\vec{j} + \left(1.80\,\mathrm{m/s^2}\right)\vec{k}$$

# Three-Dimensional Motion. Coriolis Acceleration




- With respect to the fixed frame *OXYZ* and rotating frame *Oxyz*,

$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \left(\dot{\vec{Q}}\right)_{Oxyz} + \vec{\Omega} \times \vec{Q}$$

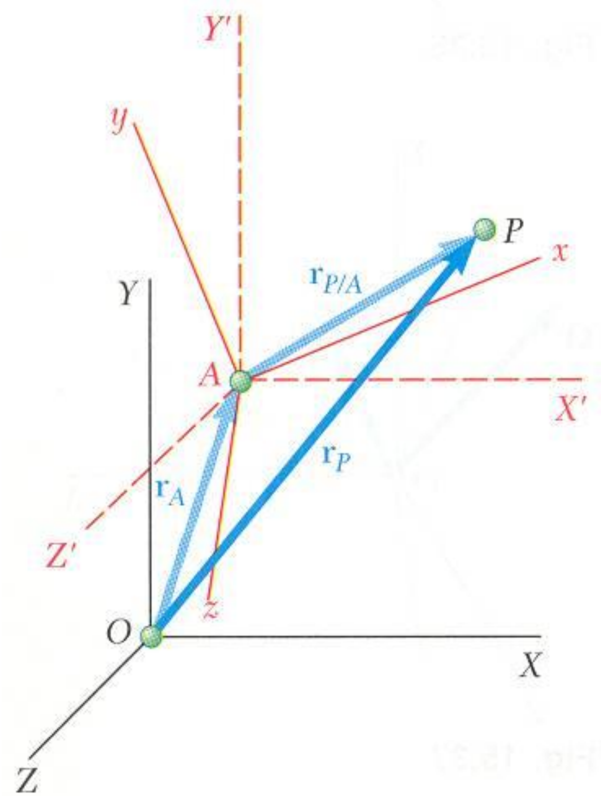
- Consider motion of particle *P* relative to a rotating frame *Oxyz* or $\mathfrak{F}$ for short. The absolute velocity can be expressed as

$$\vec{v}_P = \vec{\Omega} \times \vec{r} + \left(\dot{\vec{r}}\right)_{Oxyz}$$
$$= \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

- The absolute acceleration can be expressed as

$$\vec{a}_P = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) + 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxyz} + \left(\ddot{\vec{r}}\right)_{Oxyz}$$
$$= \vec{a}_{p'} + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$$
$$\vec{a}_c = 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxyz} = 2\vec{\Omega} \times \vec{v}_{P/\mathfrak{F}} = \text{Coriolis acceleration}$$

# Frame of Reference in General Motion




Consider:
- fixed frame *OXYZ*,
- translating frame *AX'Y'Z'*, and
- translating and rotating frame *Axyz* or $\mathscr{F}$.

- With respect to *OXYZ* and *AX'Y'Z'*,

$$\vec{r}_P = \vec{r}_A + \vec{r}_{P/A}$$

$$\vec{v}_P = \vec{v}_A + \vec{v}_{P/A}$$

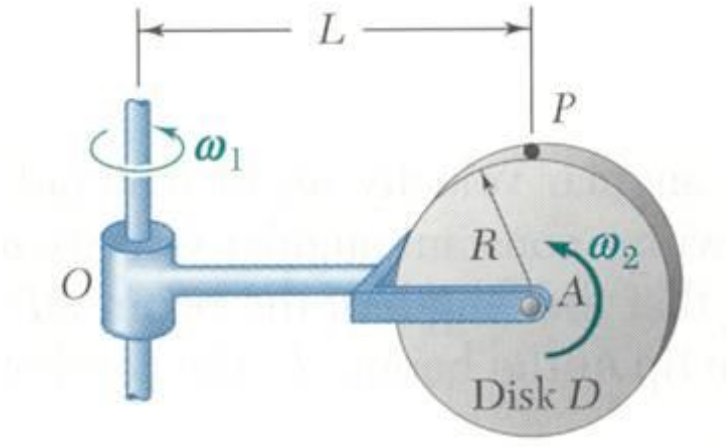
$$\vec{a}_P = \vec{a}_A + \vec{a}_{P/A}$$

- The velocity and acceleration of *P* relative to *AX'Y'Z'* can be found in terms of the velocity and acceleration of *P* relative to *Axyz*.

$$\vec{v}_P = \vec{v}_A + \vec{\Omega}\times\vec{r}_{P/A} + \left(\dot{\vec{r}}_{P/A}\right)_{Axyz} = \vec{v}_{P'} + \vec{v}_{P/\mathscr{F}}$$

$$\vec{a}_P = \vec{a}_A + \dot{\vec{\Omega}}\times\vec{r}_{P/A} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}_{P/A}\right) + 2\vec{\Omega}\times\left(\dot{\vec{r}}_{P/A}\right)_{Axyz} + \left(\ddot{\vec{r}}_{P/A}\right)_{Axyz} = \vec{a}_{P'} + \vec{a}_{P/\mathscr{F}} + \vec{a}_c$$

## Sample Problem 15.15




For the disk mounted on the arm, the indicated angular rotation rates are constant.

Determine:

- the velocity of the point $P$,
- the acceleration of $P$, and
- angular velocity and angular acceleration of the disk.

SOLUTION:

- Define a fixed reference frame $OXYZ$ at $O$ and a moving reference frame $Axyz$ or $\mathfrak{F}$ attached to the arm at $A$.
- With $P'$ of the moving reference frame coinciding with $P$, the velocity of the point $P$ is found from

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

- The acceleration of $P$ is found from

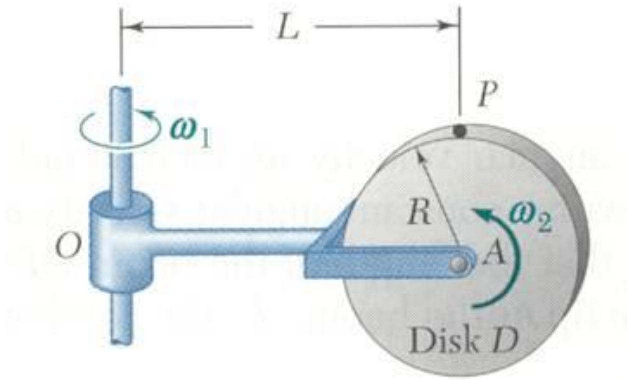
$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$$

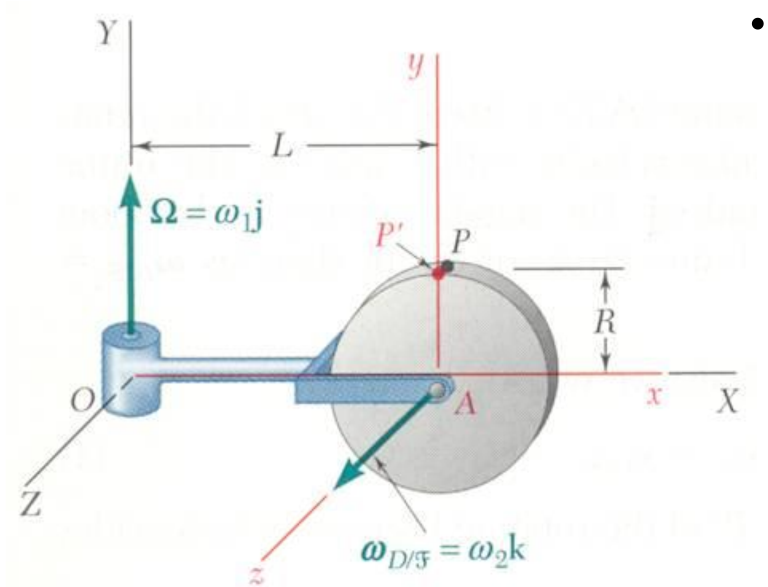
- The angular velocity and angular acceleration of the disk are

$$\vec{\omega} = \vec{\Omega} + \vec{\omega}_{D/\mathfrak{F}}$$

$$\vec{\alpha} = \left(\dot{\vec{\omega}}\right)_{\mathfrak{F}} + \vec{\Omega} \times \vec{\omega}$$

## Sample Problem 15.15




SOLUTION:

- Define a fixed reference frame *OXYZ* at *O* and a moving reference frame *Axyz* or $\mathfrak{F}$ attached to the arm at *A*.

$$\vec{r} = L\vec{i} + R\vec{j} \qquad \vec{r}_{P/A} = R\vec{j}$$

$$\vec{\Omega} = \omega_1 \vec{j} \qquad \vec{\omega}_{D/\mathfrak{F}} = \omega_2 \vec{k}$$

- With *P′* of the moving reference frame coinciding with *P*, the velocity of the point *P* is found from

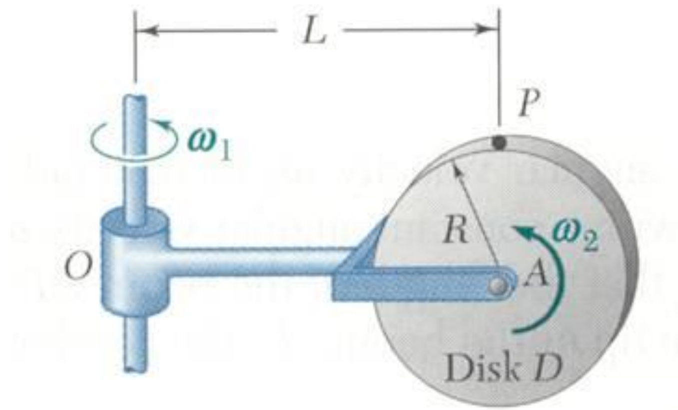
$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

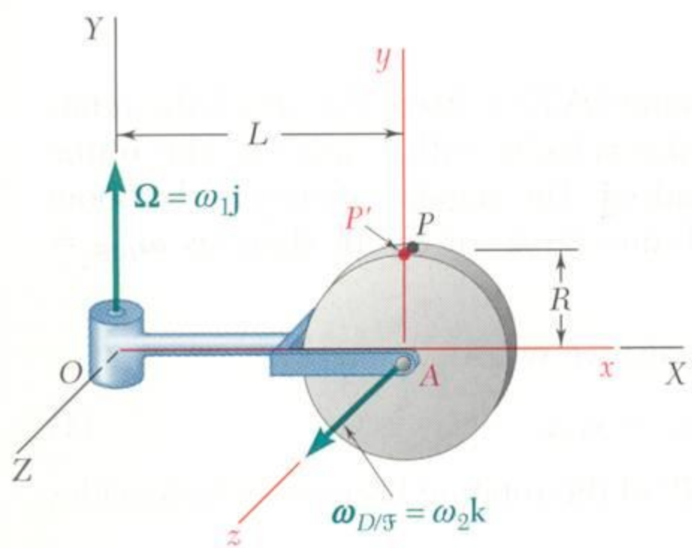
$$\vec{v}_{P'} = \vec{\Omega} \times \vec{r} = \omega_1 \vec{j} \times \left(L\vec{i} + R\vec{j}\right) = -\omega_1 L\,\vec{k}$$

$$\vec{v}_{P/\mathfrak{F}} = \vec{\omega}_{D/\mathfrak{F}} \times \vec{r}_{P/A} = \omega_2 \vec{k} \times R\vec{j} = -\omega_2 R\,\vec{i}$$

$$\boxed{\vec{v}_P = -\omega_2 R\,\vec{i} - \omega_1 L\,\vec{k}}$$

## Sample Problem 15.15




- The acceleration of $P$ is found from

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$$

$$\vec{a}_{P'} = \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) = \omega_1\vec{j}\times\left(-\omega_1 L\vec{k}\right) = -\omega_1^2 L\vec{i}$$

$$\vec{a}_{P/\mathfrak{F}} = \vec{\omega}_{D/\mathfrak{F}}\times\left(\vec{\omega}_{D/\mathfrak{F}}\times\vec{r}_{P/A}\right)$$

$$= \omega_2\vec{k}\times\left(-\omega_2 R\,\vec{i}\right) = -\omega_2^2 R\,\vec{j}$$

$$\vec{a}_c = 2\vec{\Omega}\times\vec{v}_{P/\mathfrak{F}}$$

$$= 2\omega_1\vec{j}\times\left(-\omega_2 R\,\vec{i}\right) = 2\omega_1\omega_2 R\vec{k}$$

$$\boxed{\vec{a}_P = -\omega_1^2 L\,\vec{i} - \omega_2^2 R\vec{j} + 2\omega_1\omega_2 R\vec{k}}$$

- Angular velocity and acceleration of the disk,

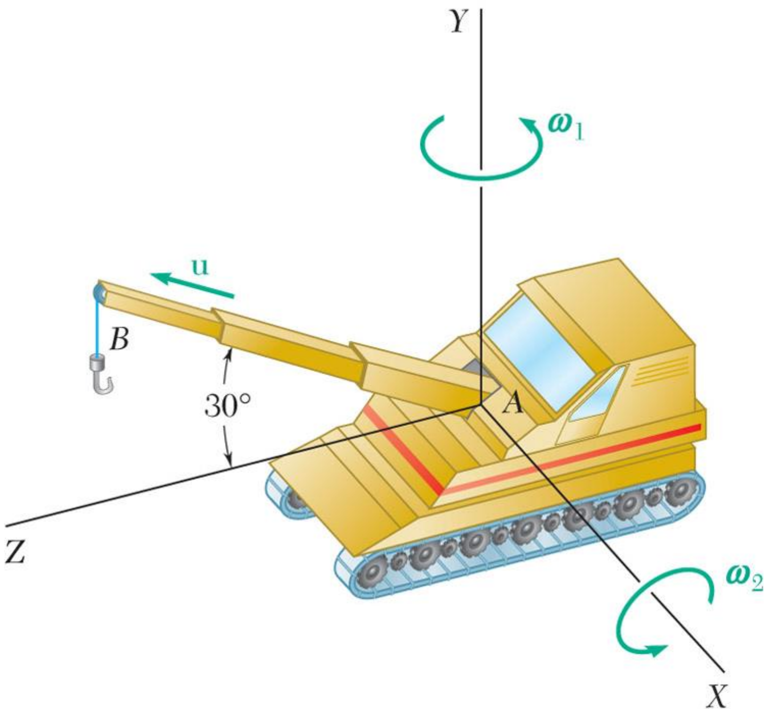
$$\vec{\omega} = \vec{\Omega} + \vec{\omega}_{D/\mathfrak{F}} \qquad \boxed{\vec{\omega} = \omega_1\vec{j} + \omega_2\vec{k}}$$

$$\vec{\alpha} = \left(\dot{\vec{\omega}}\right)_{\mathfrak{F}} + \vec{\Omega}\times\vec{\omega}$$

$$= \omega_1\vec{j}\times\left(\omega_1\vec{j} + \omega_2\vec{k}\right) \qquad \boxed{\vec{\alpha} = \omega_1\omega_2\vec{i}}$$

## Group Problem Solving

The crane shown rotates at the constant rate $\omega_1$= 0.25 rad/s; simultaneously, the telescoping boom is being lowered at the constant rate $\omega_2$= 0.40 rad/s. Knowing that at the instant shown the length of the boom is 6 m and is increasing at the constant rate *u*= 0.5 m/s determine the acceleration of Point *B*.




SOLUTION:

- Define a moving reference frame *Axyz* or $\mathcal{F}$ attached to the arm at *A*.
- The acceleration of *P* is found from

$$\vec{a}_B = \vec{a}_{B'} + \vec{a}_{B/\mathbf{F}} + \vec{a}_c$$

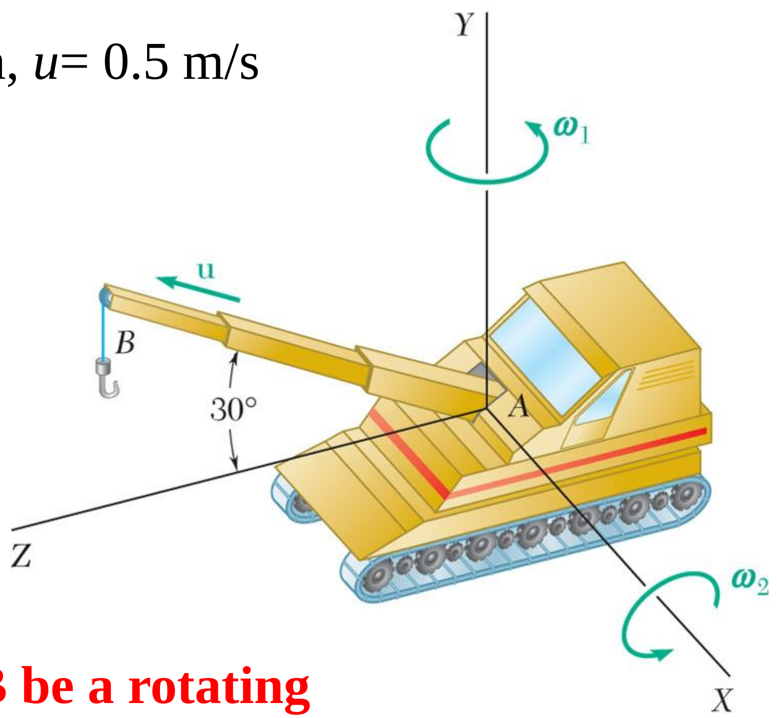
- The angular velocity and angular acceleration of the disk are

$$\vec{w} = \vec{W} + \vec{w}_{B/\mathbf{F}}$$

$$\vec{a} = \left(\dot{\vec{w}}\right)_{\mathbf{F}} + \vec{W} \times \vec{w}$$

## Group Problem Solving

Given: $\omega_1$= 0.25 rad/s, $\omega_2$= -0.40 rad/s. L= 6 m, *u*= 0.5 m/s
Find: $\boldsymbol{a_B}$.




**Equation of overall acceleration of B**

$$\vec{a}_D = \dot{\vec{W}}\times\vec{r} + \vec{W}\times\left(\vec{W}\times\vec{r}\right) + 2\vec{W}\times\left(\dot{\vec{r}}\right)_{Oxy} + \left(\ddot{\vec{r}}\right)_{Oxy}$$

**Do any of the terms go to zero?**

$$\vec{a}_D = \dot{\vec{W}}\times\vec{r} + \vec{W}\times\left(\vec{W}\times\vec{r}\right) + 2\vec{W}\times\left(\dot{\vec{r}}\right)_{Oxy} + \left(\ddot{\vec{r}}\right)_{Oxy}$$

**Let the unextending portion of the boom AB be a rotating frame of reference. What are** $\vec{W}$ and $\dot{\vec{W}}$?

$$\boldsymbol{\Omega} = \omega_2\mathbf{i} + \omega_1\mathbf{j}$$
$$= (0.40\text{ rad/s})\mathbf{i} + (0.25\text{ rad/s})\mathbf{j}.$$

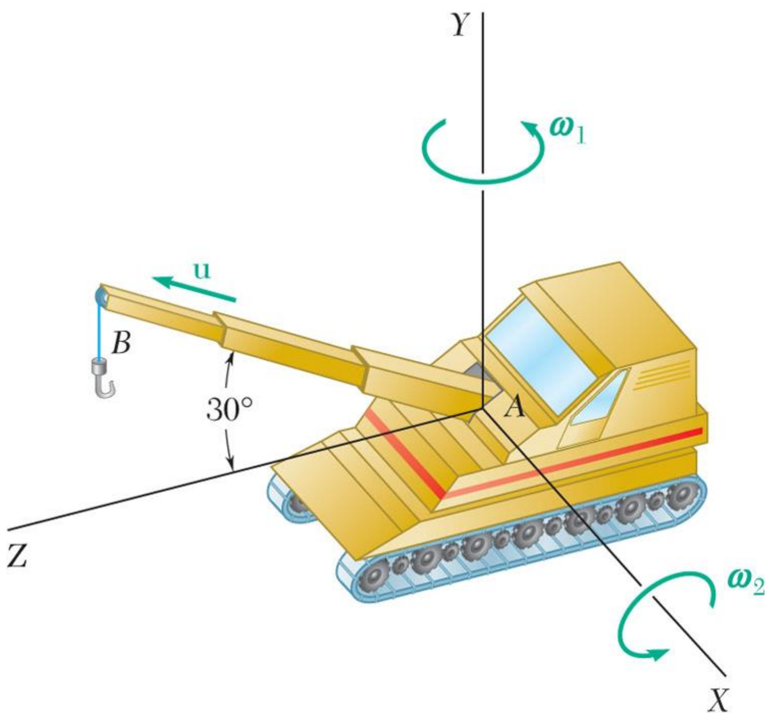
$$\dot{\vec{W}} = w_1\mathbf{j}\times w_2\mathbf{i}$$
$$= -\,w_1 w_2\mathbf{k}$$
$$= -\,(0.10\text{ rad/s}^2)\mathbf{k}.$$

## Group Problem Solving

$$\vec{a}_D = \dot{\vec{W}}\times\vec{r} + \vec{W}\times\left(\vec{W}\times\vec{r}\right) + 2\vec{W}\times\left(\dot{\vec{r}}\right)_{Oxy} + \cancel{\left(\ddot{\vec{r}}\right)_{Oxy}}$$

**Determine the position vector $\mathbf{r}_{B/A}$**

$$\mathbf{r}_{B/A} = \mathbf{r}_B$$
$$= (6\text{ m})(\sin 30°\,\mathbf{j} + \cos 30°\,\mathbf{k})$$
$$= (3\text{ m})\mathbf{j} + (3\sqrt{3}\text{ m})\mathbf{k}$$




**Find $\dot{\vec{W}} \times \vec{r}$**

$$\dot{\boldsymbol{\Omega}}\times\mathbf{r}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -0.10 \\ 0 & 3 & 3\sqrt{3} \end{vmatrix} = (0.3\text{ m/s}^2)\mathbf{i}$$

**Find $\vec{W}\times\left(\vec{W}\times\vec{r}\right)$**

$$\boldsymbol{\Omega}\times\left[\boldsymbol{\Omega}\times\mathbf{r}_B\right] = (0.40\mathbf{i}+0.25\mathbf{j})\times\left[(0.40\mathbf{i}+0.25\mathbf{j})\times(3\mathbf{j}+3\sqrt{3}\mathbf{k})\right]$$
$$= (0.3\text{ m/s}^2)\mathbf{i} - (0.48\text{ m/s}^2)\mathbf{j} - (1.1561\text{ m/s}^2)\mathbf{k}$$

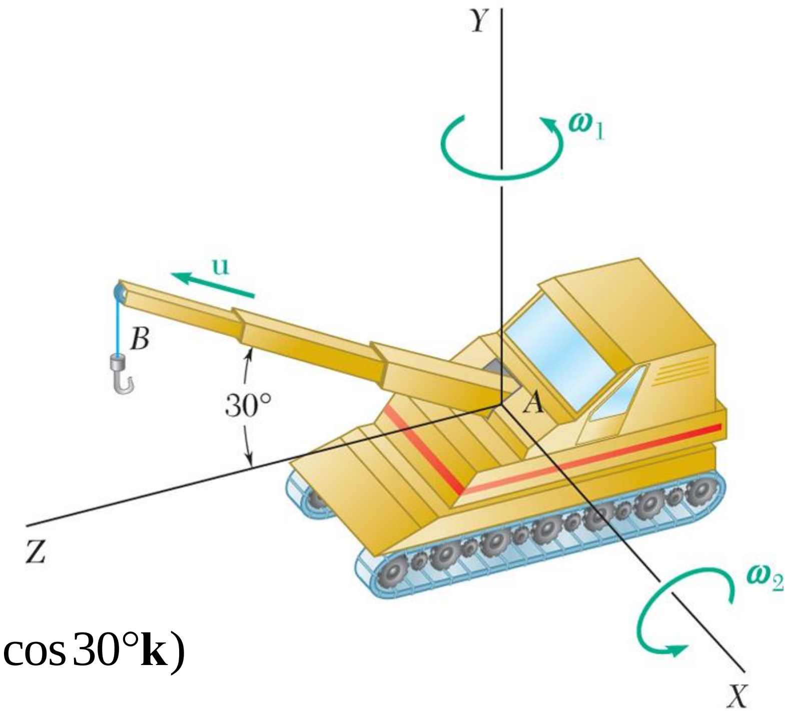
## Group Problem Solving

$$\vec{a}_D = \dot{\vec{W}} \times \vec{r} + \vec{W} \times \left(\vec{W} \times \vec{r}\right) + 2\vec{W} \times \left(\dot{\vec{r}}\right)_{Oxy} + \left(\ddot{\vec{r}}\right)_{Oxy}$$

**Determine the Coriolis acceleration – first define the relative velocity term**

$$\mathbf{v}_{B/F} = u(\sin 30°\mathbf{j} + \cos 30°\mathbf{k})$$

$$= (0.5 \text{ m/s})\sin 30°\mathbf{j} + (0.5 \text{ m/s})\cos 30°\mathbf{k}$$




**Calculate the Coriolis acceleration**

$$2\mathbf{\Omega} \times \mathbf{v}_{B/F} = (2)(0.40\mathbf{i} + 0.25\mathbf{j}) \times (0.5\sin 30°\mathbf{j} + 0.5\cos 30°\mathbf{k})$$

$$= (0.2165 \text{ m/s}^2)\mathbf{i} - (0.3464 \text{ m/s}^2)\mathbf{j} + (0.2 \text{ m/s}^2)\mathbf{k}$$

**Add the terms together**

$$\mathbf{a}_B = (0.817 \text{ m/s}^2)\mathbf{i} - (0.826 \text{ m/s}^2)\mathbf{j} - (0.956 \text{ m/s}^2)\mathbf{k}$$