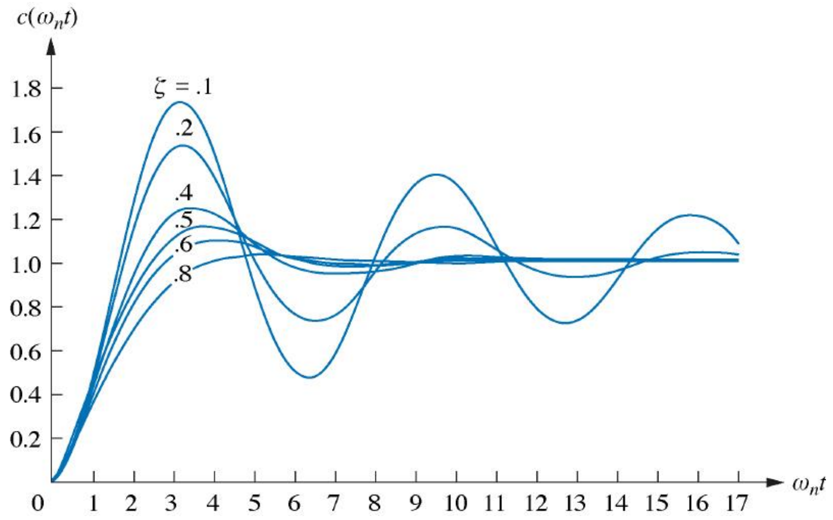
Time Response



System & Control Engineering Lab.
School of Mechanical Engineering

CHAPTER OBJECTIVES

- How to find the time response from the transfer function
- How to use poles and zeros to determine the response of a control system
- How to describe quantitatively the transient response of first- and second-order systems
- How to approximate higher-order systems as first or second order

POLES, ZEROS, AND SYSTEM RESPONSE

- POLES OF A TRANSFER FUNCTION
- ZEROS OF A TRANSFER FUNCTION
- FIRST-ORDER SYSTEMS
- SECOND-ORDER SYSTEMS
- SYSTEM RESPONSE WITH ADDITIONAL POLES
- SYSTEM RESPONSE WITH ZEROS

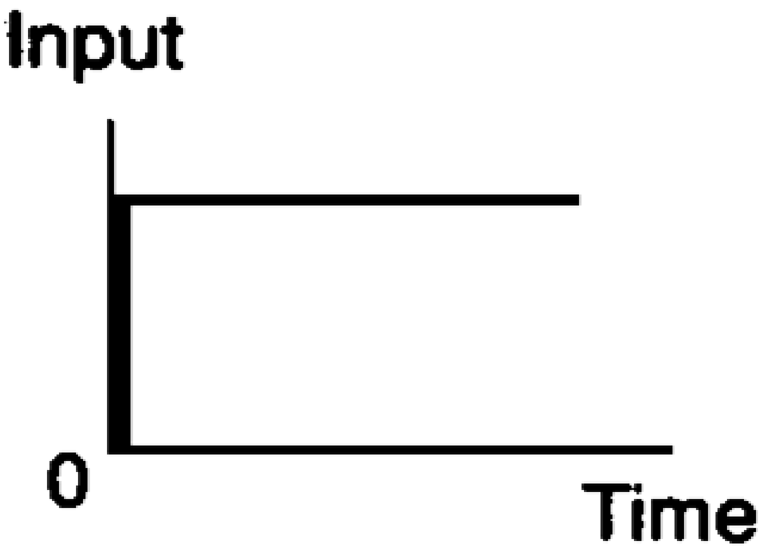
The Performance of Control Systems

- Test input signals
- Transient response and Steady-state response
- Absolute stability, relative stability
- Steady-state error

Typical Test Signal

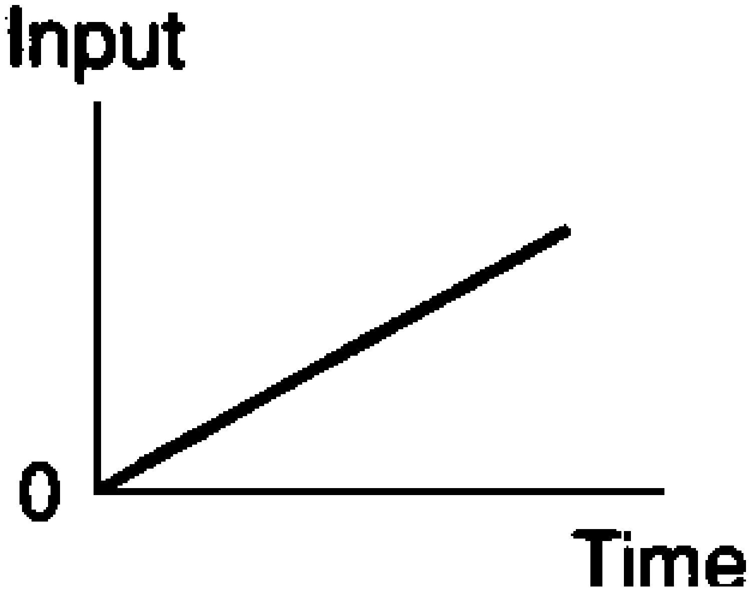
- Step functions
- Ramp functions
- Acceleration functions
- Impulse functions
- Sinusoidal functions
- White noise signals

Typical Test Signal



Step functions

$$f(t) = 1(t); F(s) = \frac{1}{s}$$

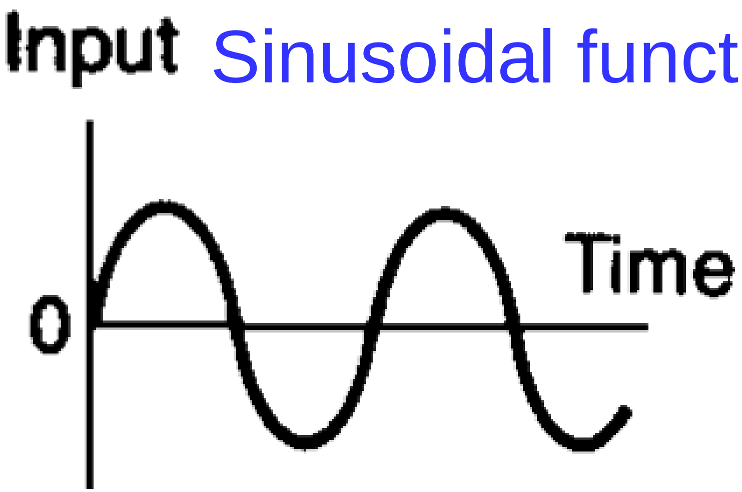


Ramp functions

$$f(t) = t; F(s) = \frac{1}{s^2}$$

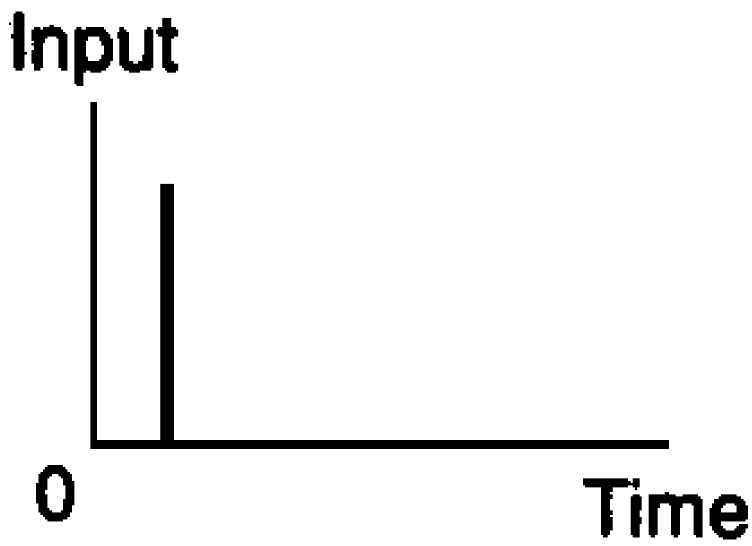
Typical Test Signal

Sinusoidal functions



$$f(t) = \sin \omega t; F(s) = \frac{\omega}{s^2 + \omega^2}$$

$$f(t) = \cos \omega t; F(s) = \frac{s}{s^2 + \omega^2}$$



Impulse functions

$$\text{Unit impulse} = \delta(\text{t}); F(s) = 1$$

POLES AND ZEROS OF A FIRST-ORDER SYSTEM: AN EXAMPLE

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5} \tag{4.1}$$

where

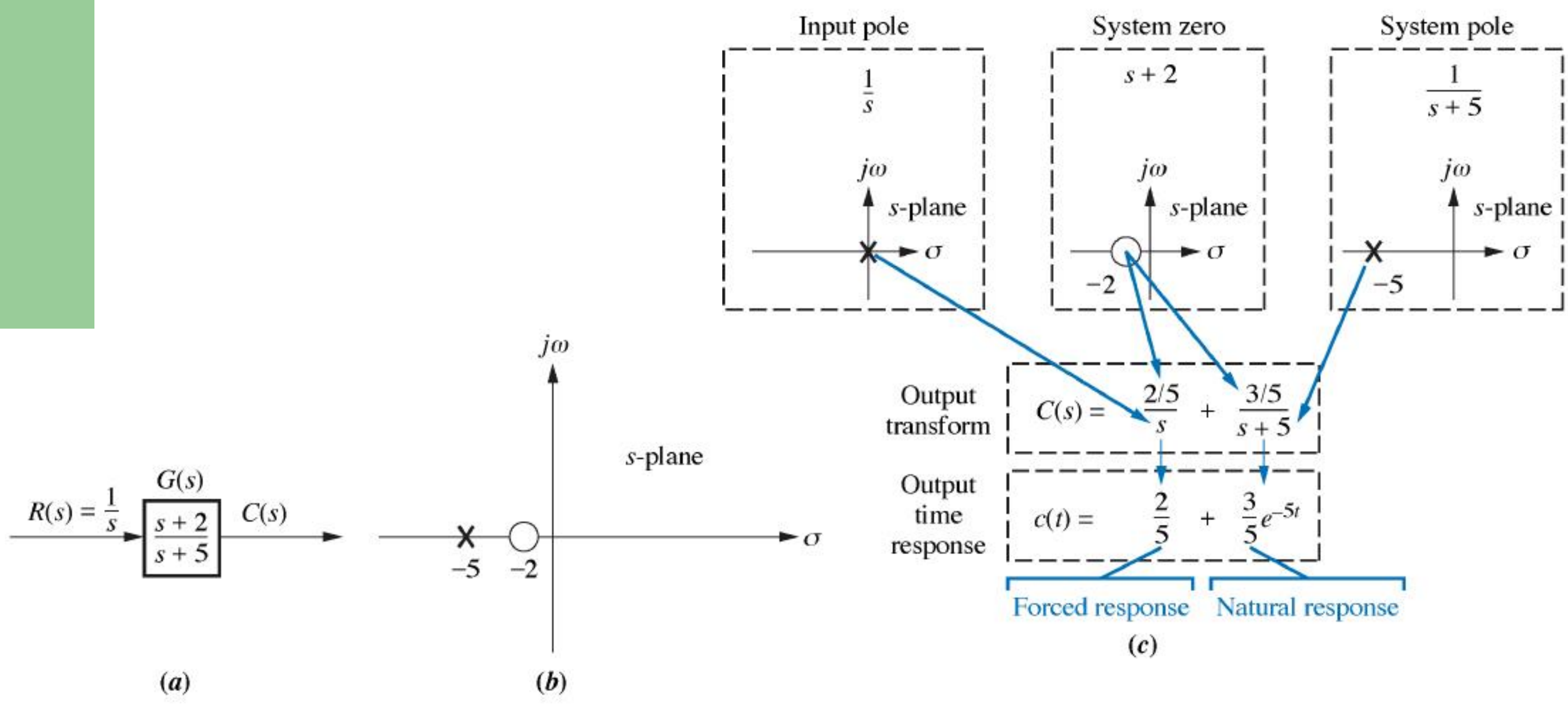
$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s \to 0} = \frac{2}{5}$$

$$B = \left.\frac{(s+2)}{s}\right|_{s \to -5} = \frac{3}{5}$$

Thus,

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t} \tag{4.2}$$

POLES AND ZEROS OF A FIRST-ORDER SYSTEM: AN EXAMPLE



POLES AND ZEROS OF A FIRST-ORDER SYSTEM: AN EXAMPLE

Evaluating response using poles

Problem: Given the system of Figure 4.3, write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.

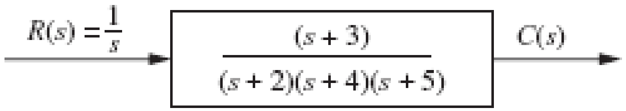


FIGURE 4.3 System for Example 4.1

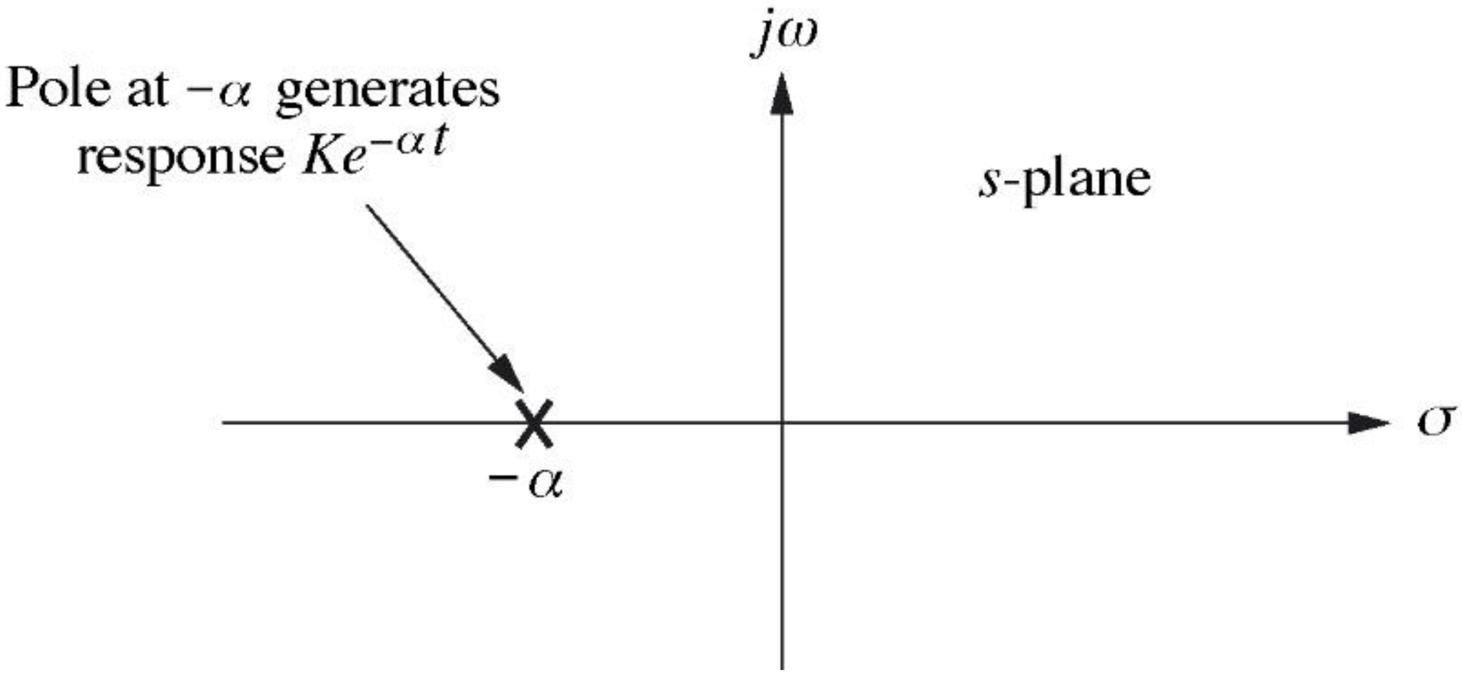
SOLUTION: By inspection, each system pole generates an exponential as part of the natural response. The input's pole generates the forced response. Thus,

$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+5)}}_{\text{Natural response}} \tag{4.3}$$

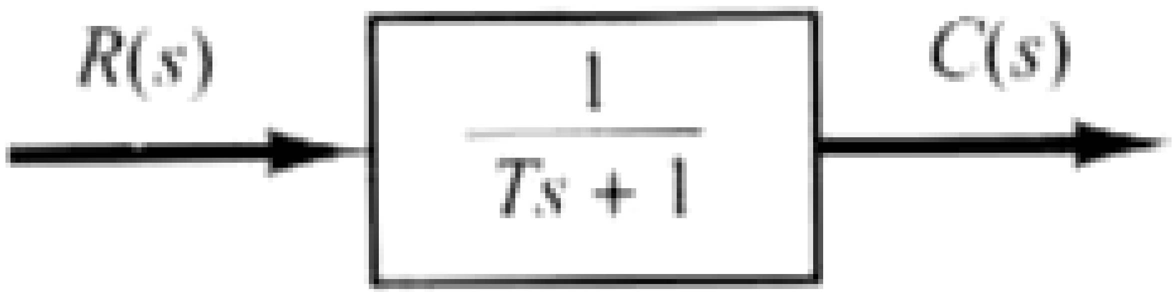
Taking the inverse Laplace transform, we get

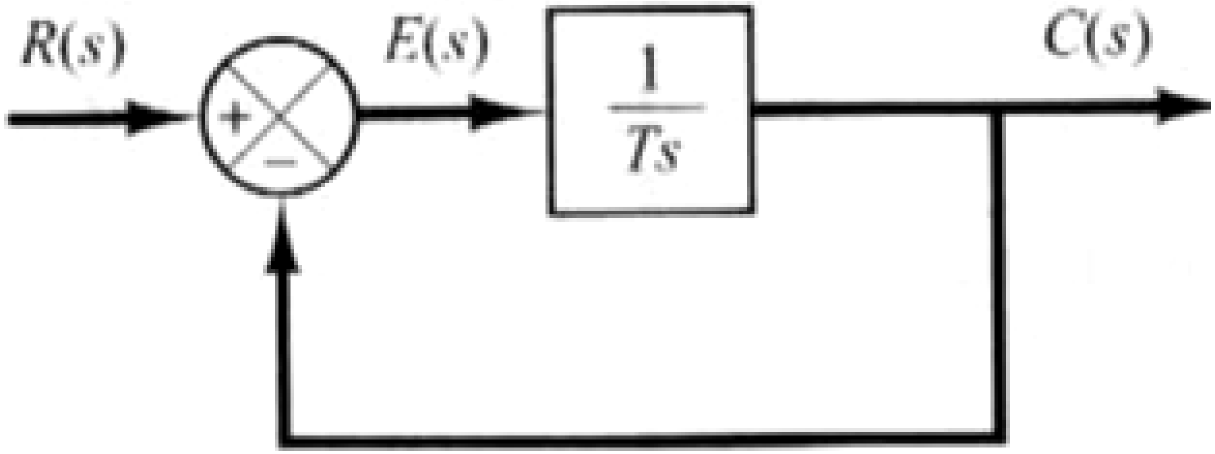
$$c(t) \equiv \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}} \tag{4.4}$$

POLES AND ZEROS OF A FIRST-ORDER SYSTEM: AN EXAMPLE

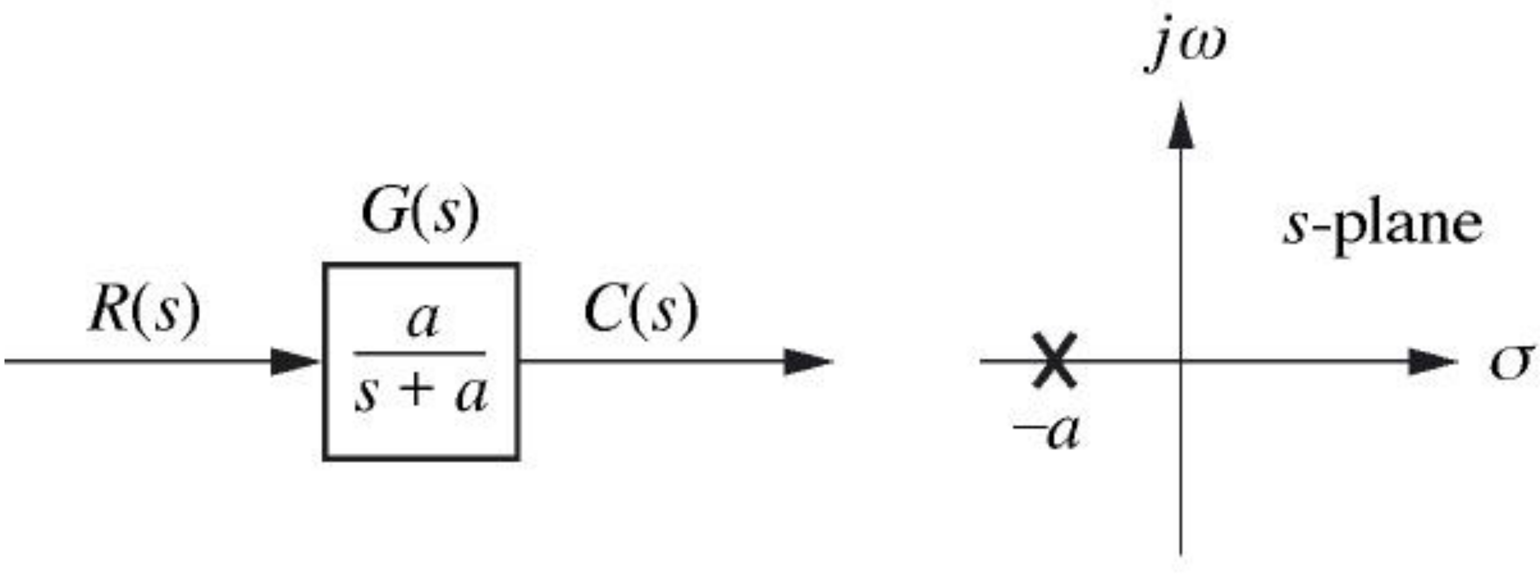


FIRST-ORDER SYSTEMS





FIRST-ORDER SYSTEMS



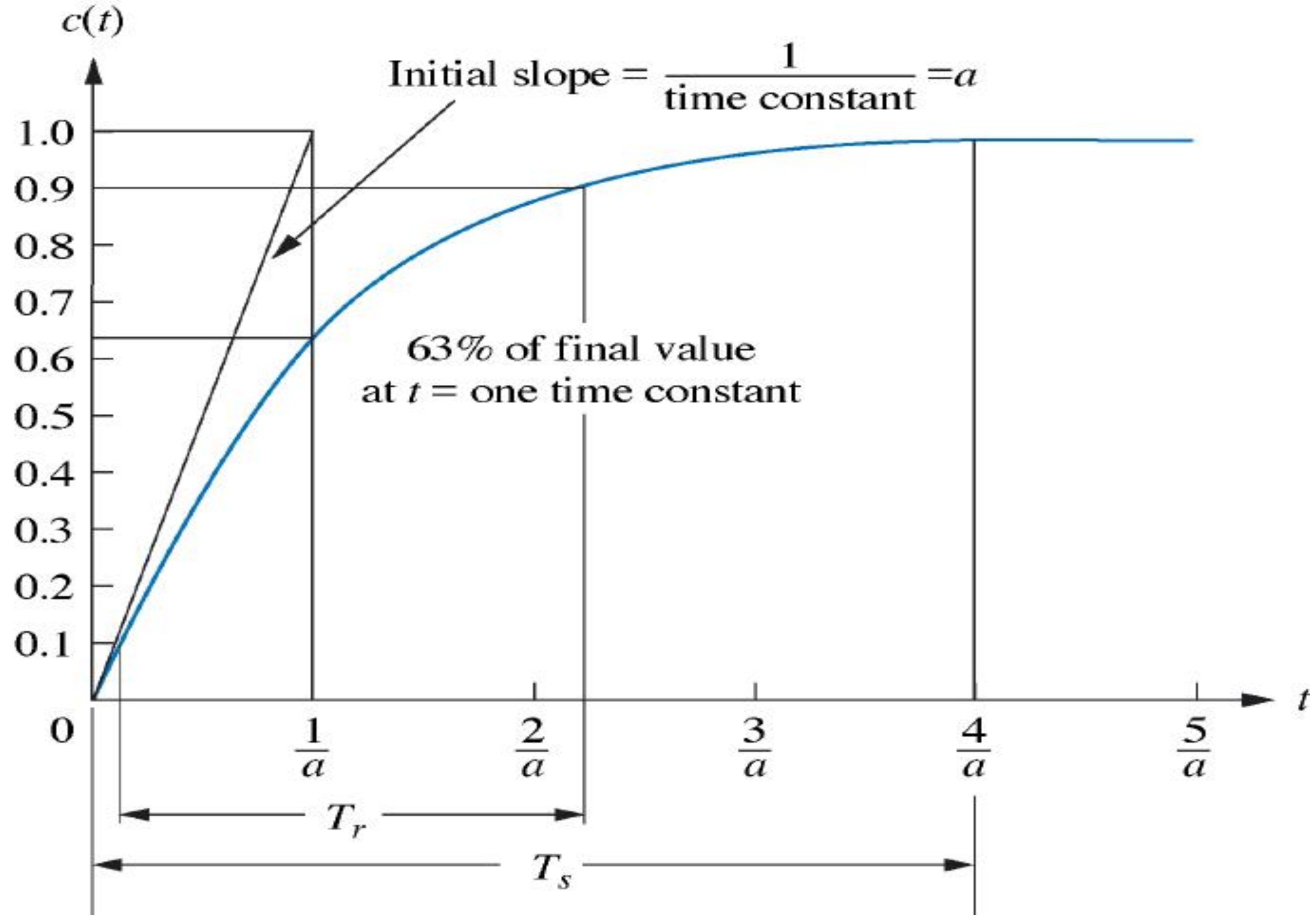
(a) (b)

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

Taking the inverse transform, the step response is given by

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

FIRST-ORDER SYSTEMS



FIRST-ORDER SYSTEMS

- TIME CONSTANT
- RISE TIME, Tr
- SETTLING TIME, Ts

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

$$T_s = \frac{4}{a}$$

Unit-Step Response of First-Order Systems

Transfer function of the first-order system

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1} \rightarrow C(s) = \frac{1}{Ts+1} R(s)$$

For unit-step function $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s}$$

Unit-Step Response of First-Order Systems

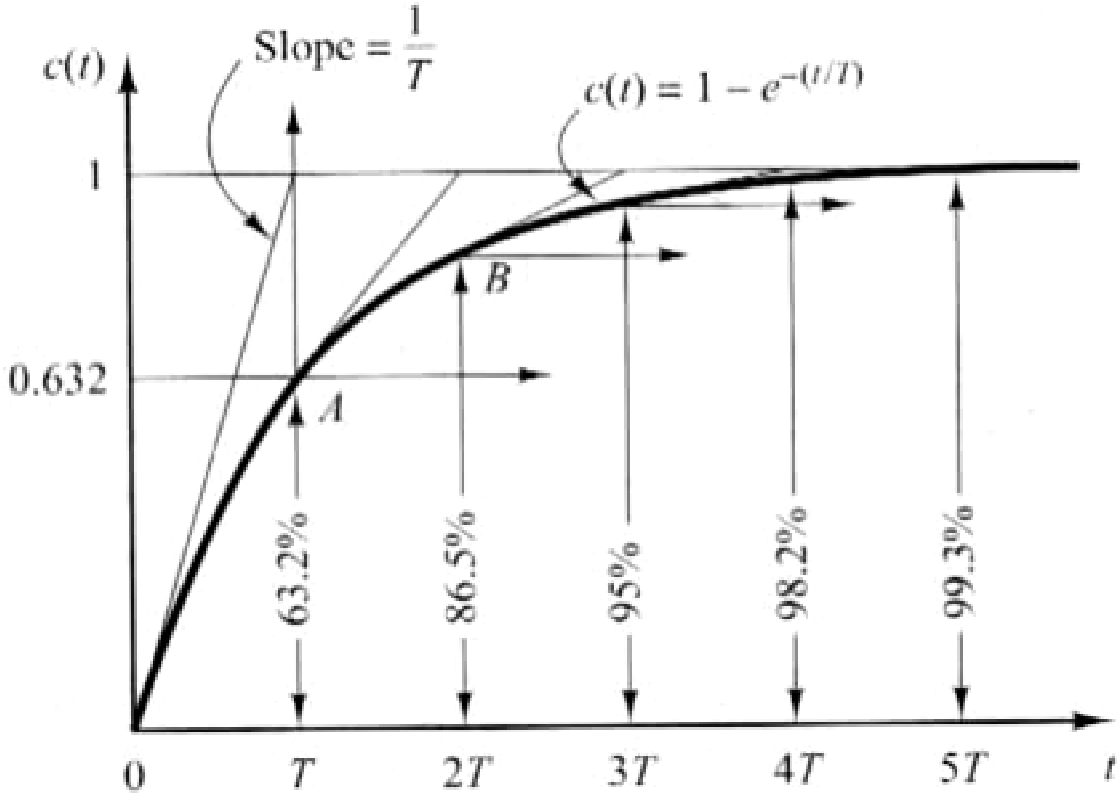
Inverse Laplace transform

$$c(t) = 1 - e^{-t/T} \quad \text{for } t \geq 0$$

C(t) = Force response + Free response

and t=T $$c(t) = 1 - e^{-1} = 0.632$$

Unit-Step Response of First-Order Systems



Unit-Step response

Second-Order Systems

In transfer function (Laplace Transform)

$$X_0(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} F_i(s)$$

$$\frac{X_0(s)}{F_i(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = G(s)$$

Second-Order Systems

1. Overdamped $\zeta > 1$
2. Critically damped $\zeta = 1$
3. Undamped $\zeta = 0$
4. Underdamped $0 < \zeta < 1$

Second-Order Systems

1. *Overdamped responses*

   Poles: Two real at $-\sigma_1, -\sigma_2$

   Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

2. *Underdamped responses*

   Poles: Two complex at $-\sigma_d \pm j\omega_d$

   Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$

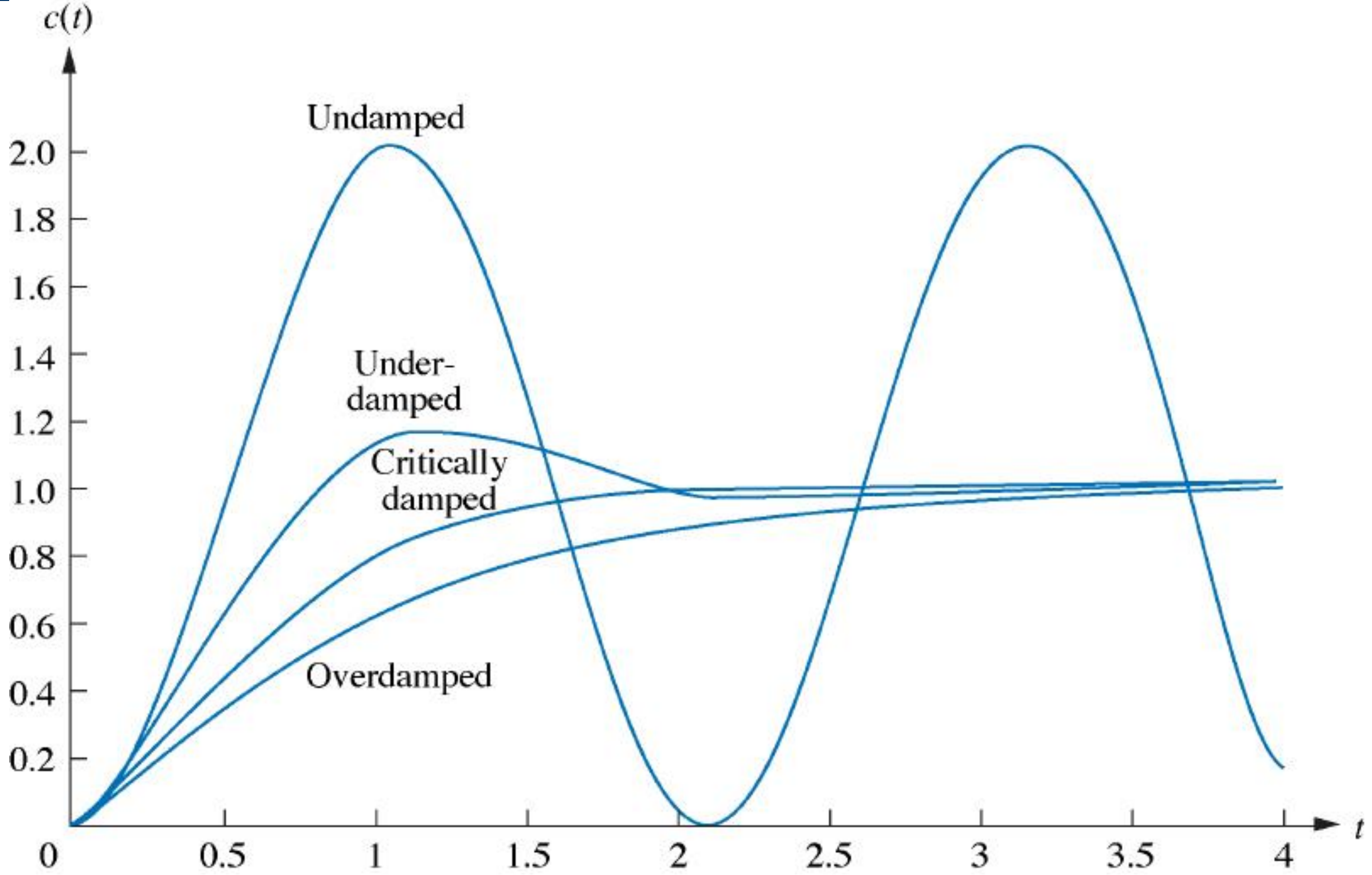
3. *Undamped responses*

   Poles: Two imaginary at $\pm j\omega_1$

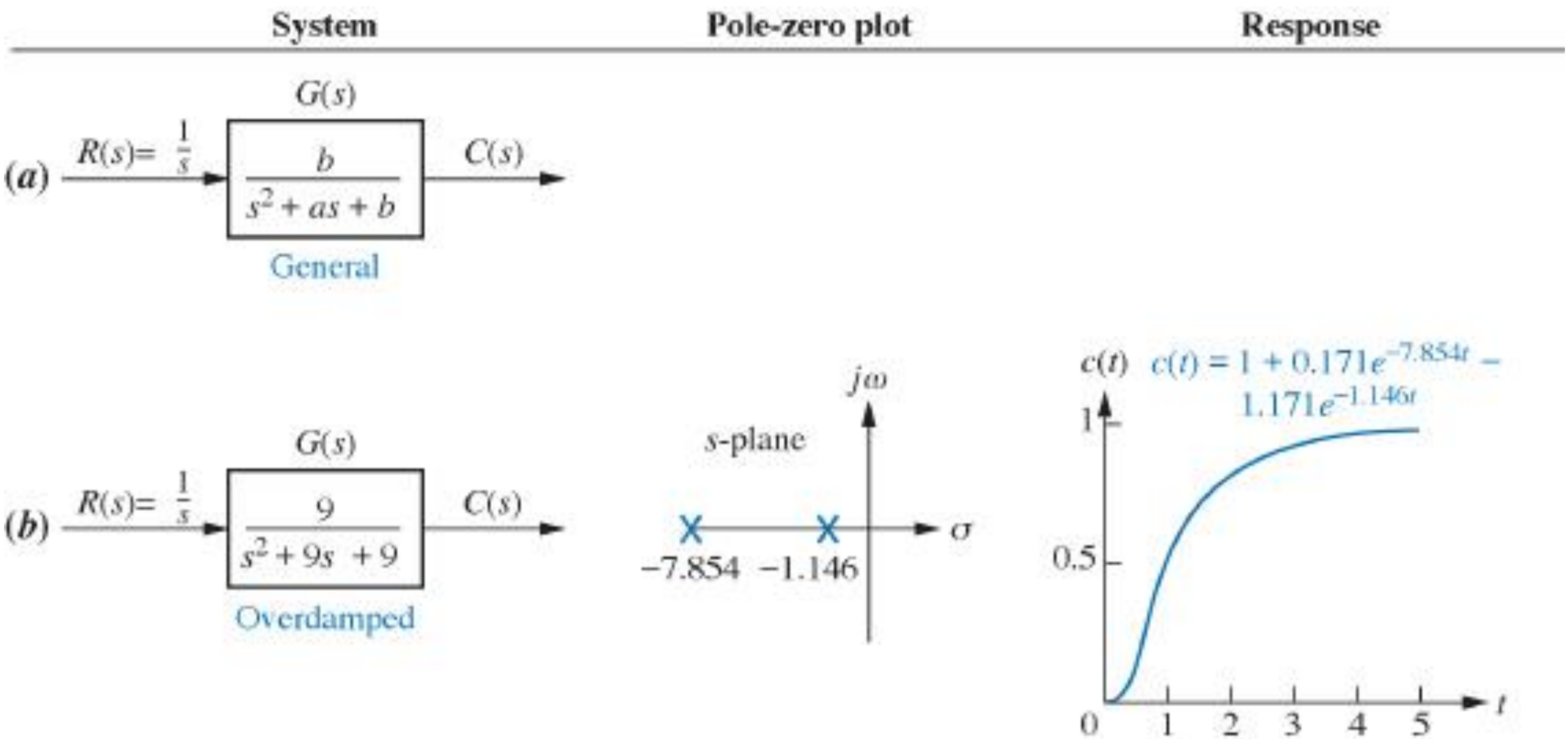
   Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

$$c(t) = A\cos(\omega_1 t - \phi)$$

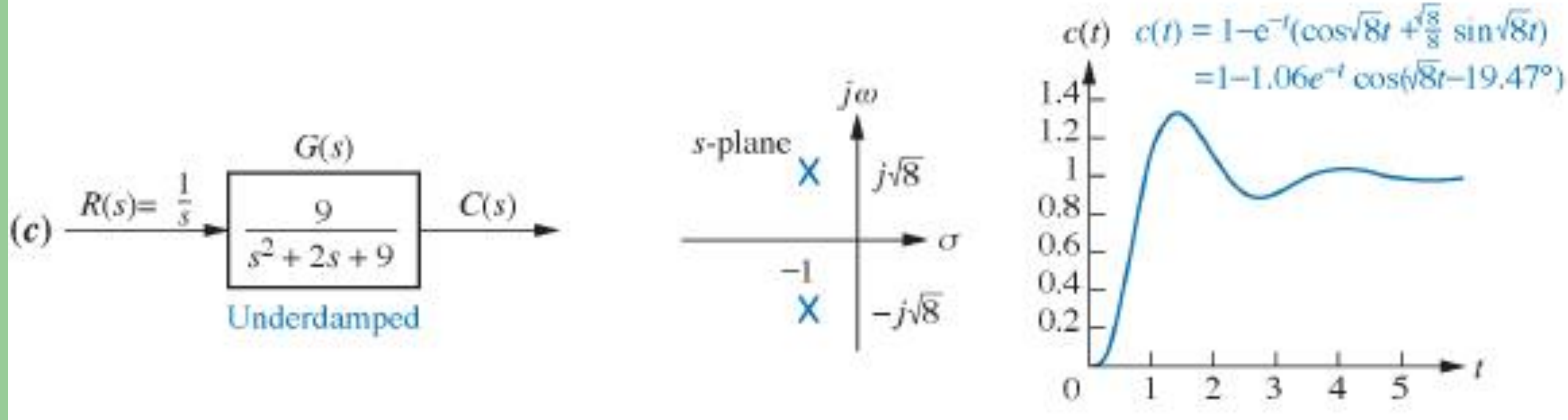
Second-Order Systems



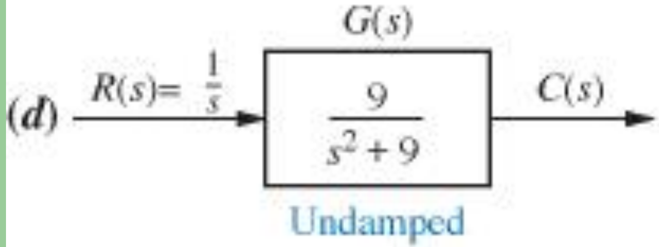
Second-Order Systems

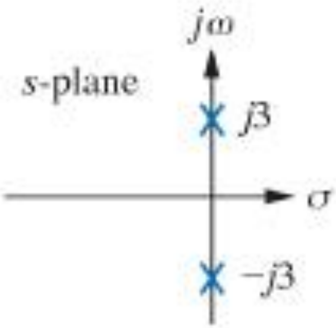


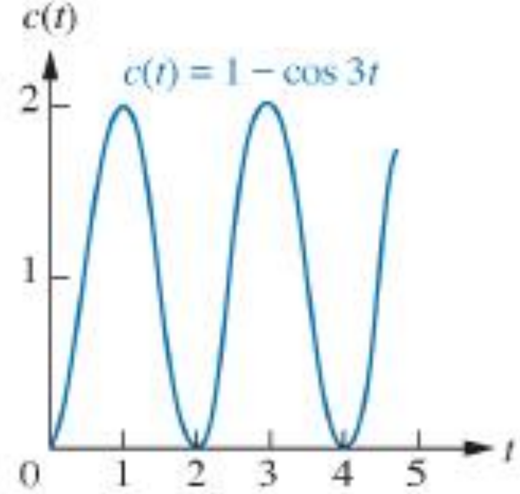
Second-Order Systems



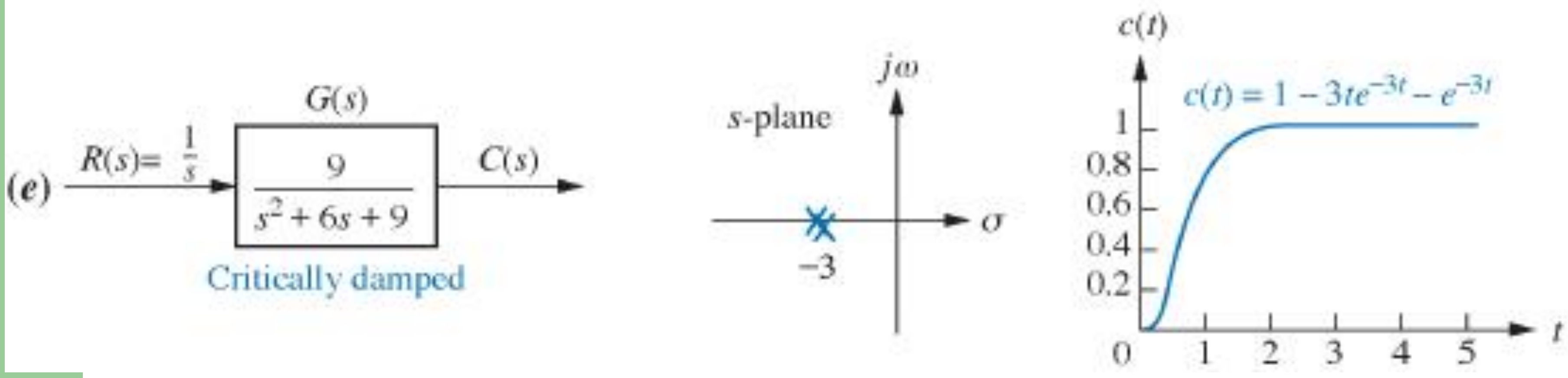
Second-Order Systems



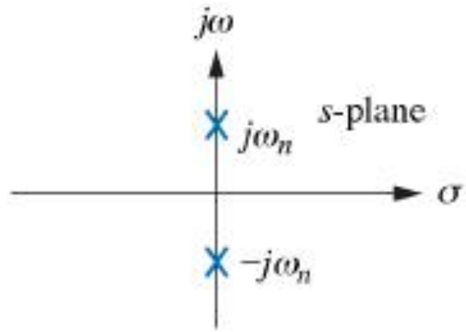
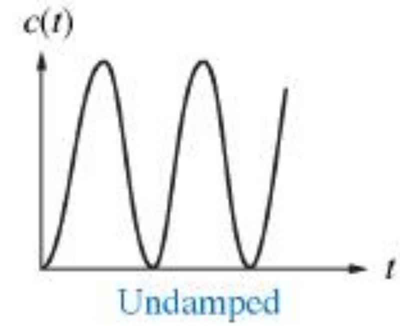
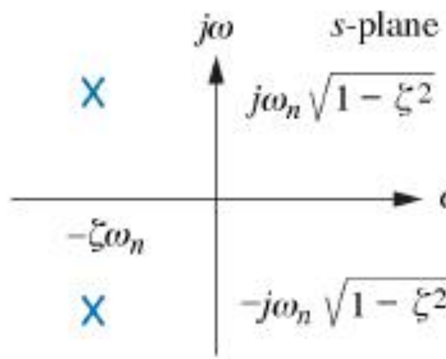
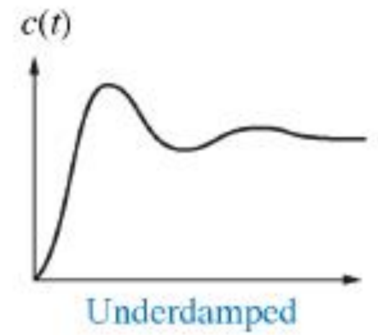




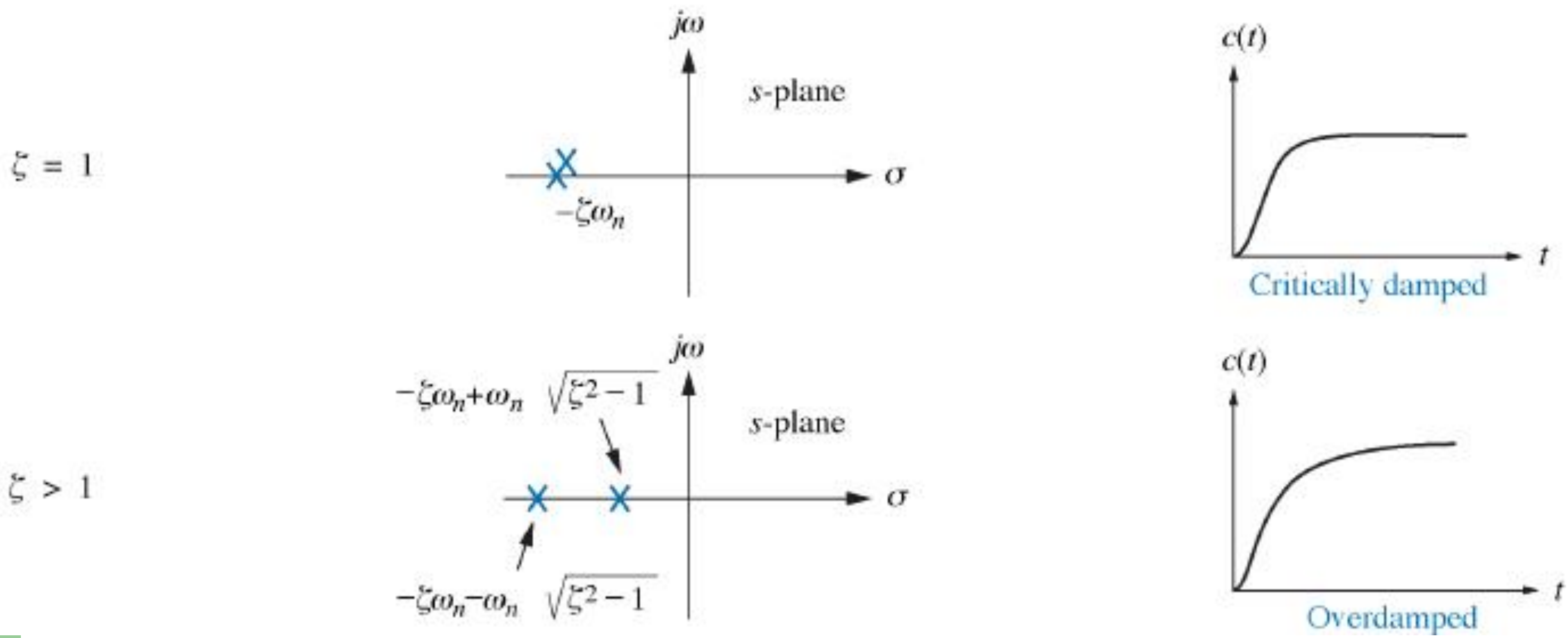
Second-Order Systems



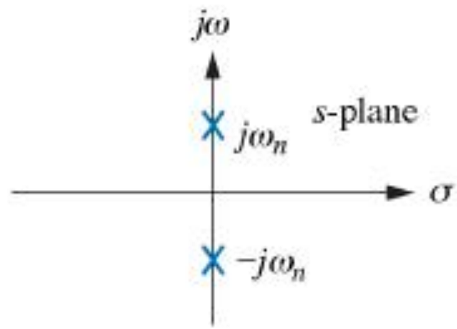
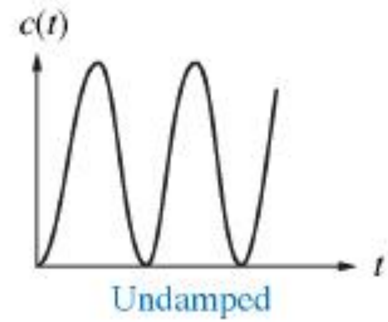
Second-Order Systems

ζ	Poles	Step response
0		
$0 < \zeta < 1$		

Second-Order Systems



Second-Order Systems

ζ	Poles	Step response
0		

Undamped Second-Order Systems Response

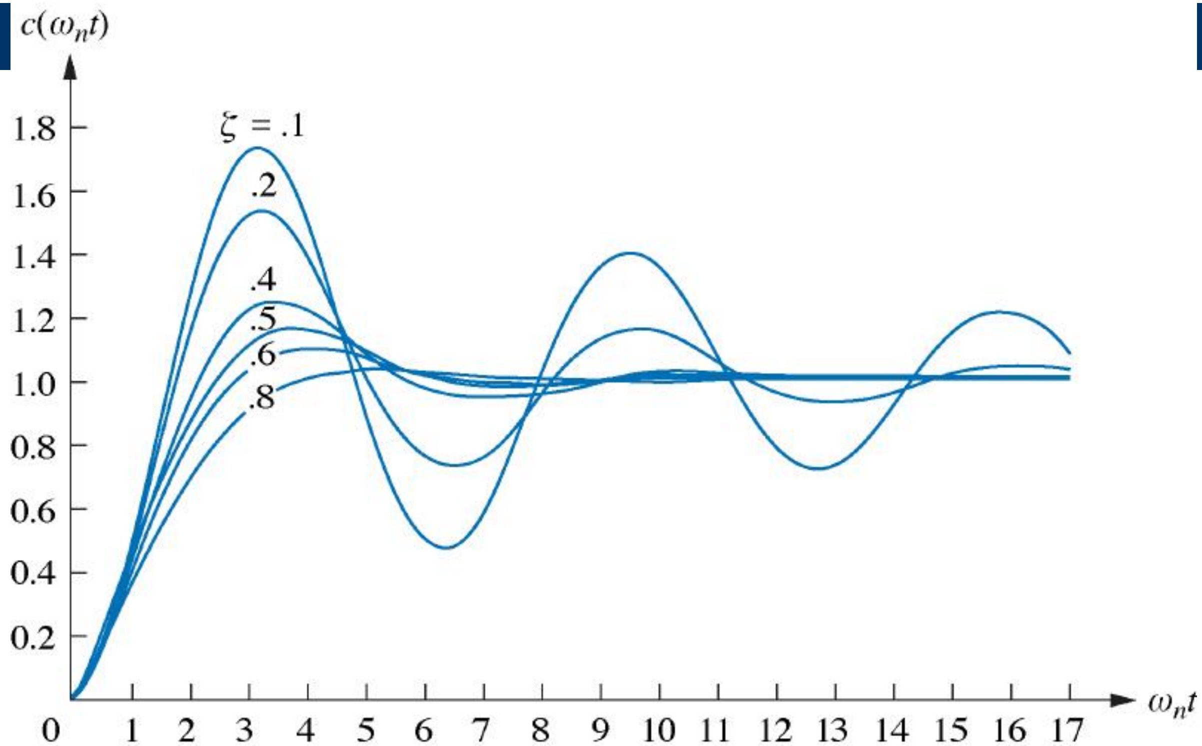
Let us begin by finding the step response for the general second-order system of Eq. (4.22). The transform of the response, $C(s)$, is the transform of the input times the transfer function, or

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (4.26)$$

where it is assumed that $\zeta < 1$ (the underdamped case). Expanding by partial fractions, using the methods described in Section 2.2, Case 3, yields

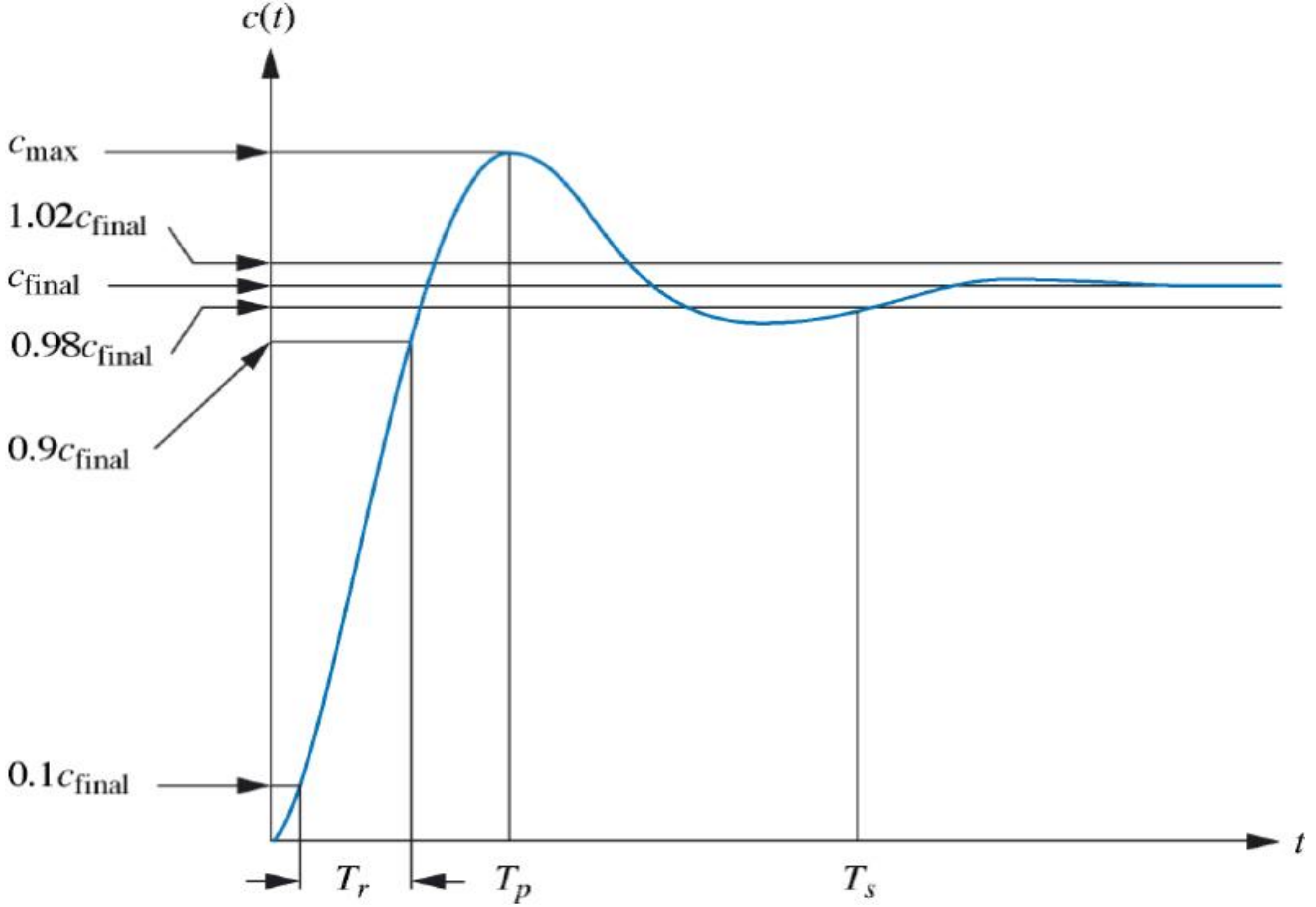
$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \qquad (4.27)$$

Undamped Second-Order Systems Response



$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n \sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_n \sqrt{1-\zeta^2}\,t\right)$$
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1-\zeta^2}\,t - \phi\right) \qquad (4.28)$$

Undamped Second-Order Systems Response



Undamped Second-Order Systems Response

1. *Rise time*, T_r. The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
2. *Peak time*, T_P. The time required to reach the first, or maximum, peak.
3. *Percent overshoot*, *%OS*. The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.
4. *Settling time*, T_s. The time required for the transient's damped oscillations to reach and stay within ±2% of the steady-state value.

Undamped Second-Order Systems Response

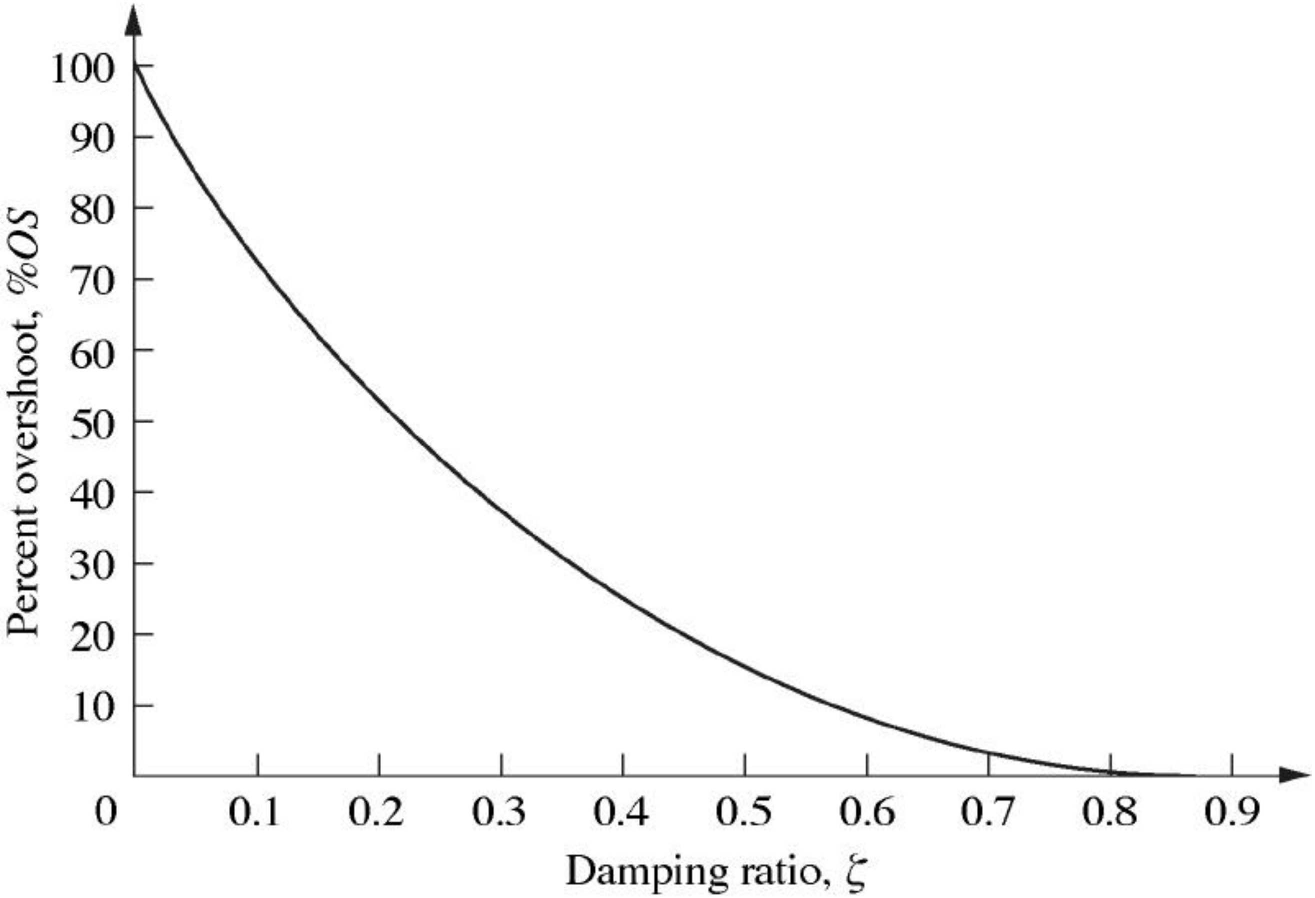
$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

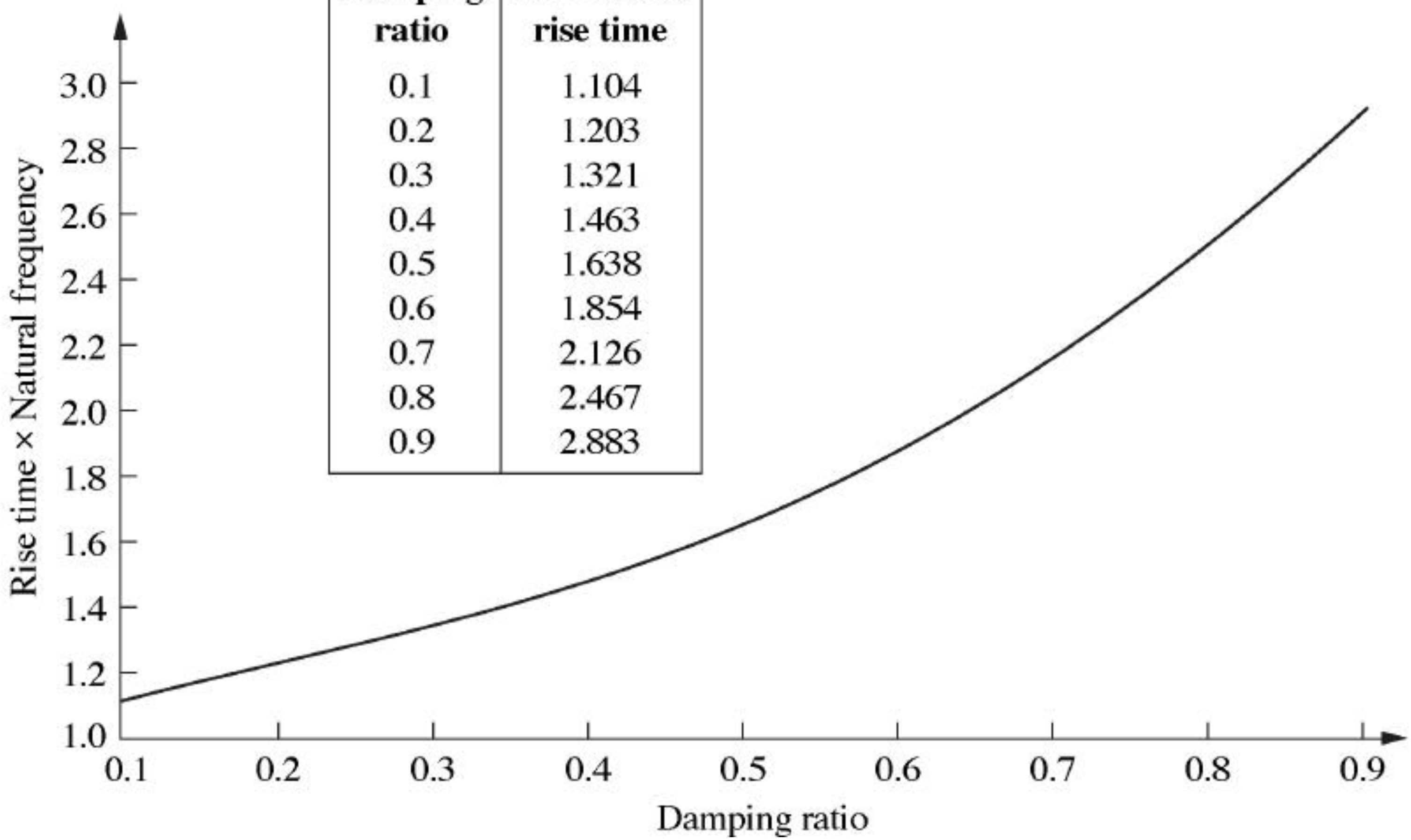
$$T_s = \frac{4}{\zeta\omega_n}$$

Undamped Second-Order Systems Response

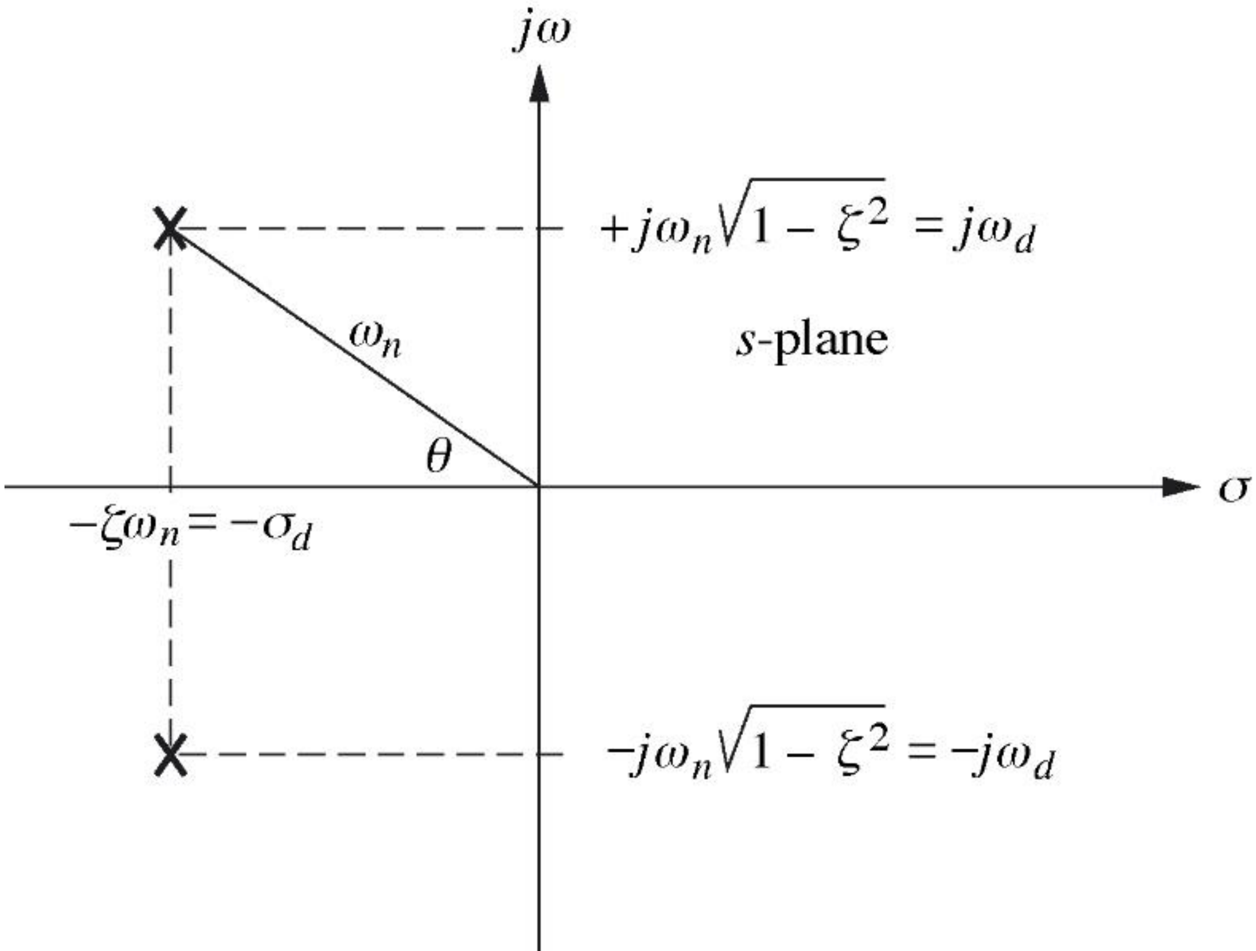


Undamped Second-Order Systems Response

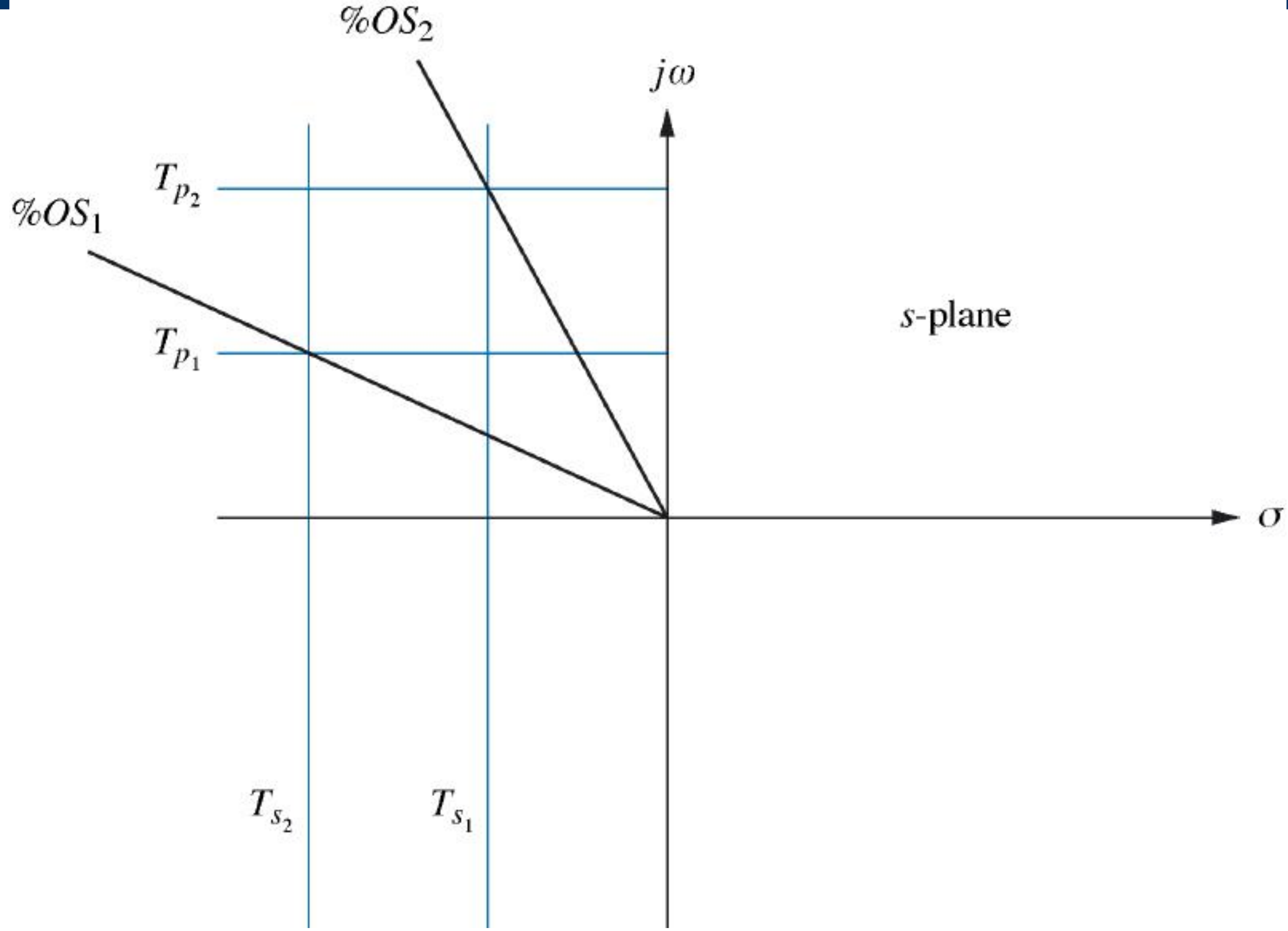
Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883



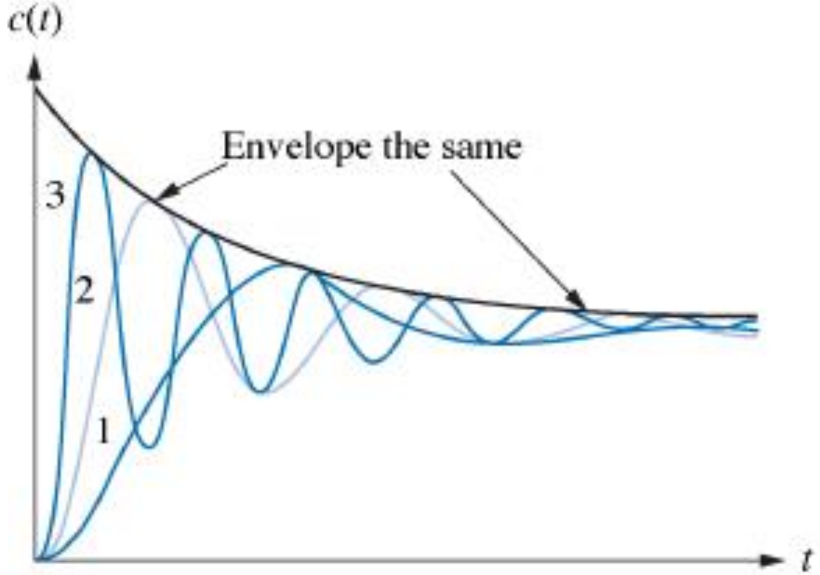
Undamped Second-Order Systems Response



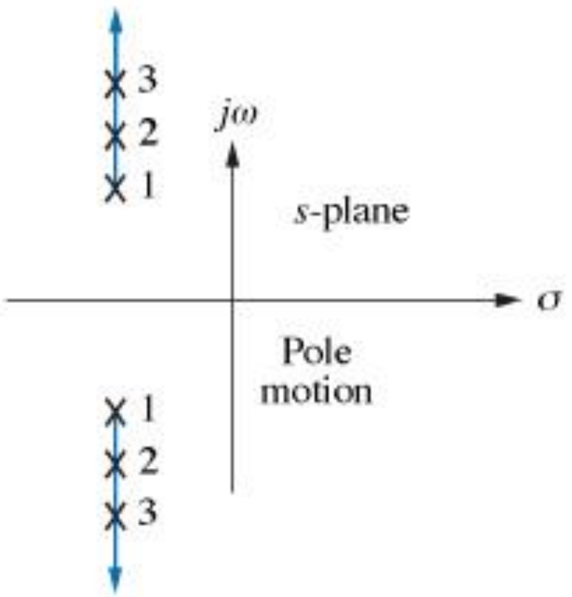
Undamped Second-Order Systems Response



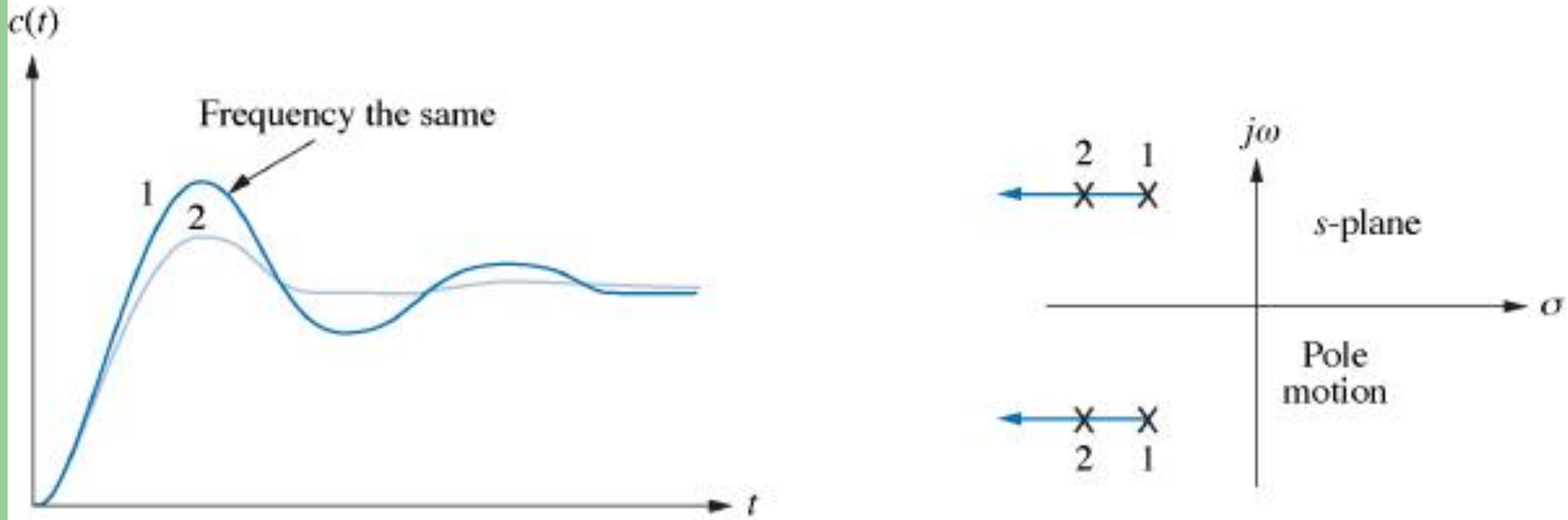
Undamped Second-Order Systems Response



(a)

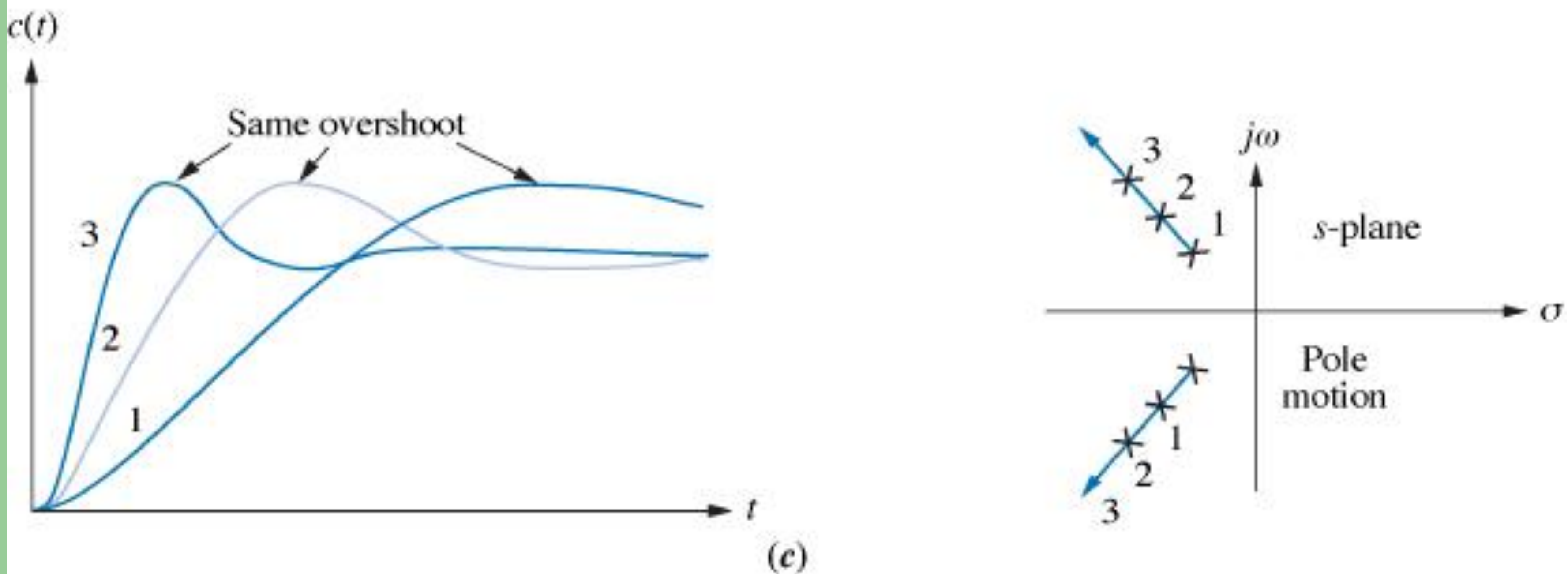


Undamped Second-Order Systems Response

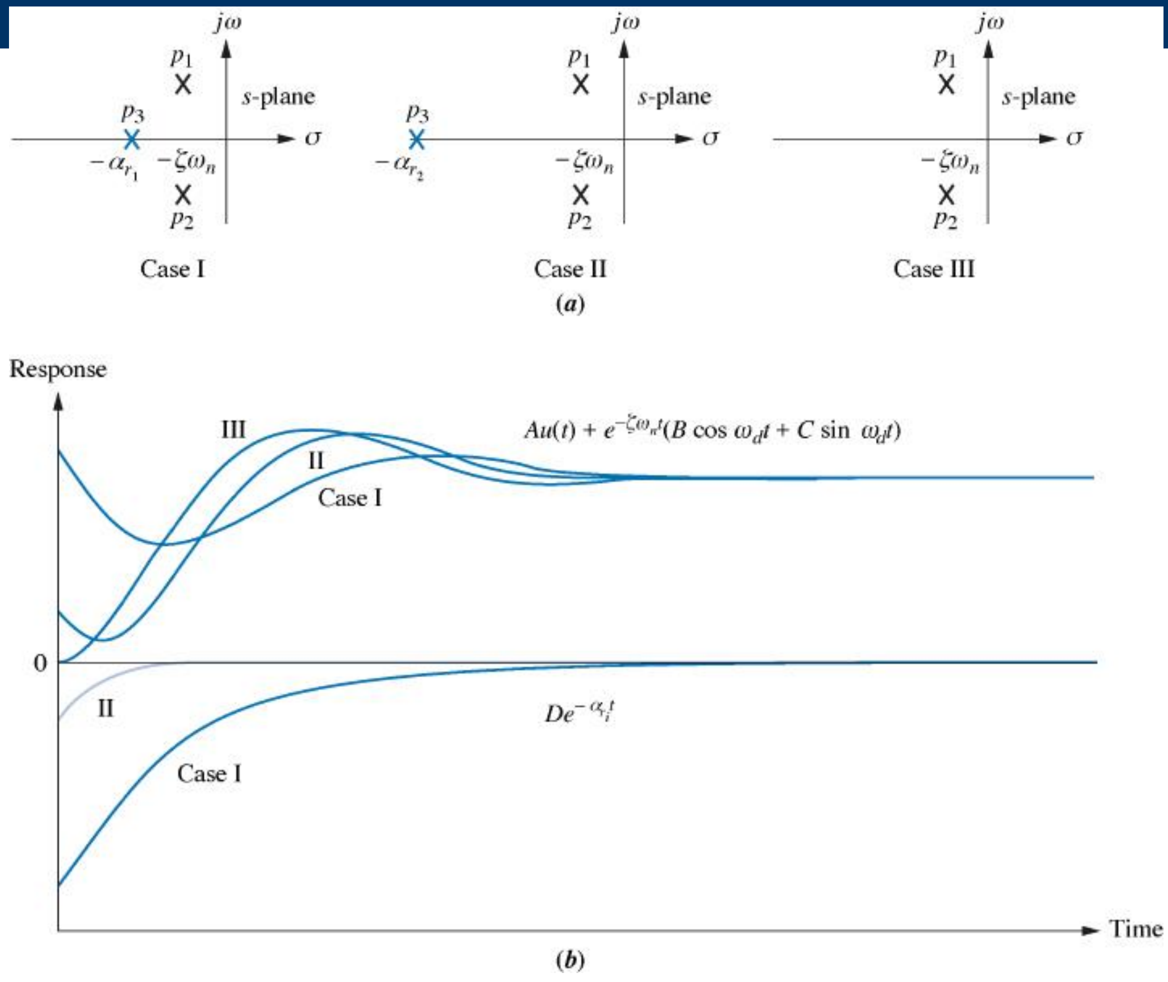


(b)

Undamped Second-Order Systems Response



SYSTEM RESPONSE WITH ADDITIONAL POLES



SYSTEM RESPONSE WITH ADDITIONAL ZEROS