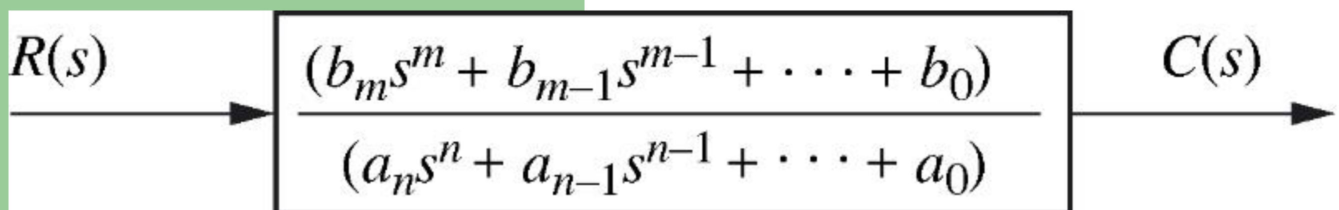
Modeling in the Frequency Domain



System & Control Engineering Lab.
School of Mechanical Engineering

CHAPTER OBJECTIVES

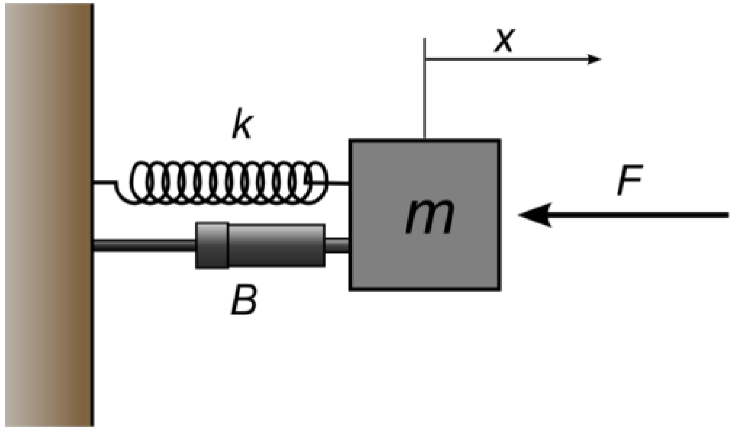
- Review the Laplace transform
- Learn how to find a mathematical model, called a transfer function, for linear, time-invariant electrical, mechanical, and electromechanical systems
- Learn how to linearize a nonlinear system in order to find the transfer function

Mechanical System

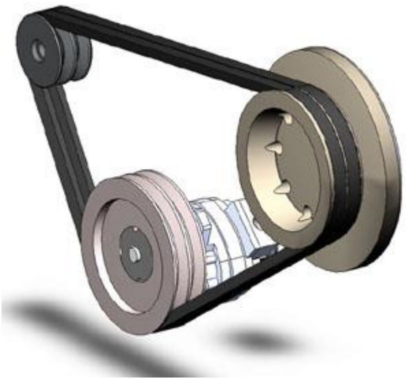
- **Translational Mechanical System**
- **Rotational Mechanical System**
- **Mechanical Linkages**

Mechanical Systems

- Translational
  - Linear Motion



- Rotational
  - Rotational Motion



Elements of Mechanical Systems

Building block	Describing equation	Energy stored/ power dissipated
Energy storage		
Translational spring	$F = kx$	$E = \frac{1}{2}\frac{F^2}{k}$
Torsional spring	$T = k\theta$	$E = \frac{1}{2}\frac{T^2}{k}$
Mass	$F = m\frac{d^2x}{dt^2}$	$E = \frac{1}{2}mv^2$
Moment of inertia	$T = I\frac{d^2\theta}{dt^2}$	$E = \frac{1}{2}I\omega^2$
Energy dissipation		
Translational dashpot	$F = c\frac{dx}{dt}$	$P = cv^2$
Rotational damper	$T = c\frac{d\theta}{dt}$	$P = c\omega^2$

Elements of Mechanical Systems

Systems of units

Systems of units / Quantity	Absolute systems			Gravitational systems	
	Metric			Metric engineering	British engineering
	SI	mks	cgs		
Length	m	m	cm	m	ft
Mass	kg	kg	g	$\frac{kg_f\text{-}s^2}{m}$	slug $= \frac{lb_f\text{-}s^2}{ft}$
Time	s	s	s	s	s
Force	N $= \frac{kg\text{-}m}{s^2}$	N $= \frac{kg\text{-}m}{s^2}$	dyn $= \frac{g\text{-}cm}{s^2}$	kg_f	lb_f
Energy	J = N-m	J = N-m	erg = dyn-cm	kg_f-m	ft-lb_f or Btu
Power	W $= \frac{N\text{-}m}{s}$	W $= \frac{N\text{-}m}{s}$	$\frac{dyn\text{-}cm}{s}$	$\frac{kg_f\text{-}m}{s}$	$\frac{ft\text{-}lb_f}{s}$ or hp

Mechanical Systems

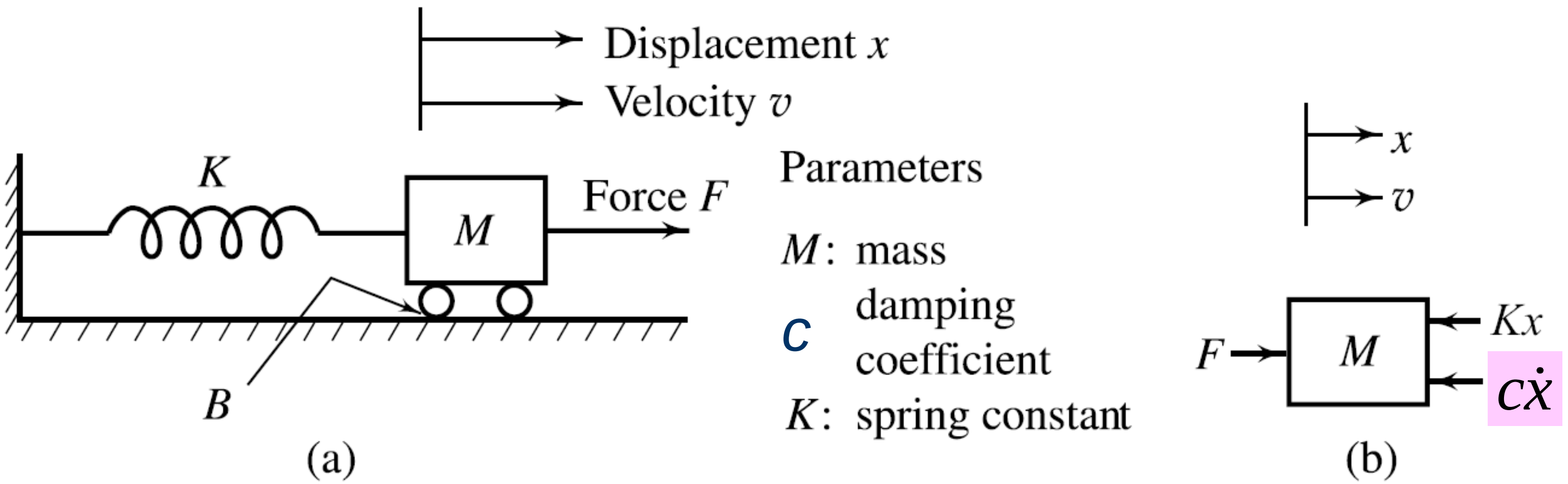


Fig. 2.3 (a) A mass-spring-damper system (b) Free-body diagram

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

Mechanical Systems

$F(t)$ – input; $x(t)$ – output

Laplace transform

$$m(s^2X(s) - sx_0 - \dot{x}_0) + c(sX(s) - x_0) + kX(s) = F(s)$$

Zero initial conditions; $x_0(t=0)=0; \dot{x}_0(t=0)=0$

$$ms^2X(s) + csX(s) + kX(s) = F(s)$$

Mechanical Systems

$F(t) - \text{input}; \;\; x(t) - \text{output}$

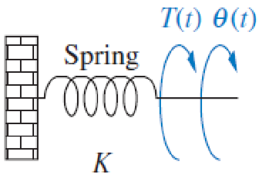
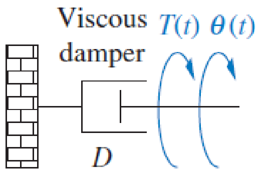
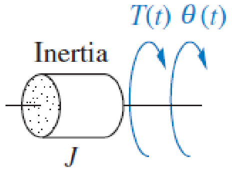
$$\left(ms^2 + cs + k\right) X(s) = F(s)$$

Transfer function

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

ROTATIONAL MECHANICAL SYSTEM TRANSFER FUNCTIONS

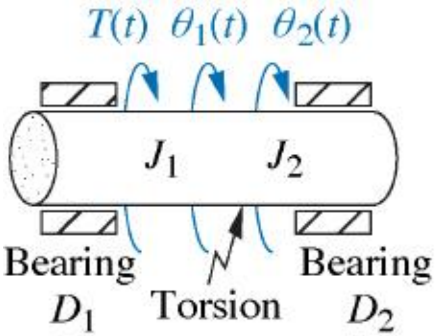
TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

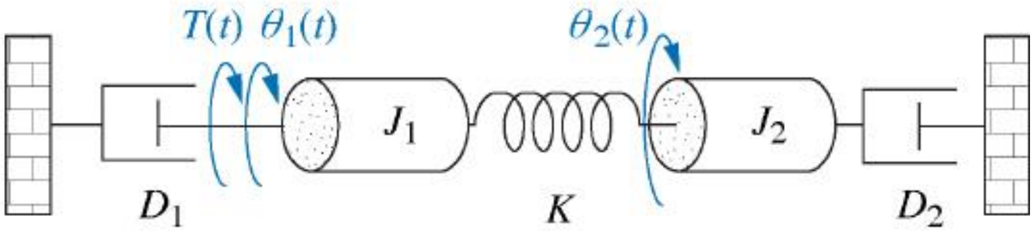
Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian). J = kg-m^2 (kilograms-meters^2 = newton-meters-seconds^2/radian).

ROTATIONAL MECHANICAL SYSTEM TRANSFER FUNCTIONS

Determine transfer function $\dfrac{\theta_2(s)}{T(s)}$



(a)



For J_1

$$\sum M = J_1\ddot{\theta}_1$$

$$-D_1\dot{\theta}_1 - K(\theta_1 - \theta_2) + T = J_1\ddot{\theta}_1$$

$$J_1\ddot{\theta}_1 + D_1\dot{\theta}_1 + K(\theta_1 - \theta_2) = T$$

$$J_1\ddot{\theta}_1 + D_1\dot{\theta}_1 + K\theta_1 - K\theta_2 = T$$

ROTATIONAL MECHANICAL SYSTEM TRANSFER FUNCTIONS

For J_2

$$\sum M = J_2\ddot{\theta}_2$$

$$-D_2\dot{\theta}_2 + K\left(\theta_1 - \theta_2\right) = J_2\ddot{\theta}_2$$

$$J_2\ddot{\theta}_2 + D_2\dot{\theta}_2 - K\left(\theta_1 - \theta_2\right) = 0$$

$$J_2\ddot{\theta}_2 + D_2\dot{\theta}_2 + K\theta_2 - K\theta_1 = 0$$

Laplace transform with zero initial conditions

$$\left(J_1 s^2 + D_1 s + K\right)\theta_1(s) - K\theta_2(s) = T(s) \qquad (1)$$

$$\left(J_2 s^2 + D_2 s + K\right)\theta_2(s) - K\theta_1(s) = 0 \qquad (2)$$

ROTATIONAL MECHANICAL SYSTEM TRANSFER FUNCTIONS

$$\theta_1(s) = \frac{J_2 s^2 + D_2 s + K}{K}\theta_1(s)$$

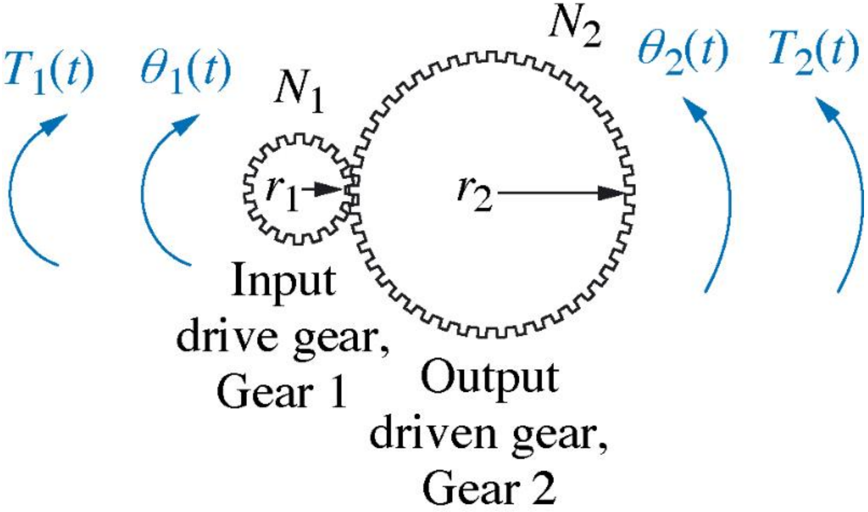
$$\frac{\left(J_1 s^2 + D_1 s + K\right)\left(J_2 s^2 + D_2 s + K\right)}{K}\theta_2(s) - K\theta_2(s) = T(s)$$

Transfer function

$$\frac{\left(J_1 s^2 + D_1 s + K\right)\left(J_2 s^2 + D_2 s + K\right) - K^2}{K}\theta_2(s) = T(s)$$

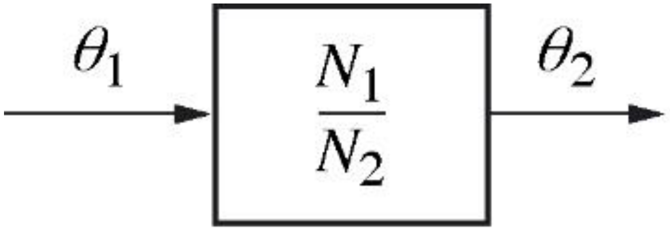
$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\left(J_1 s^2 + D_1 s + K\right)\left(J_2 s^2 + D_2 s + K\right) - K^2}$$

TRANSFER FUNCTIONS FOR SYSTEMS WITH GEARS

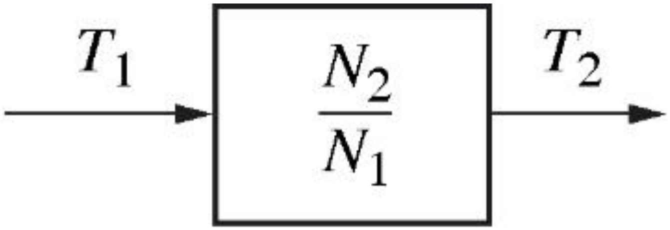


$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$



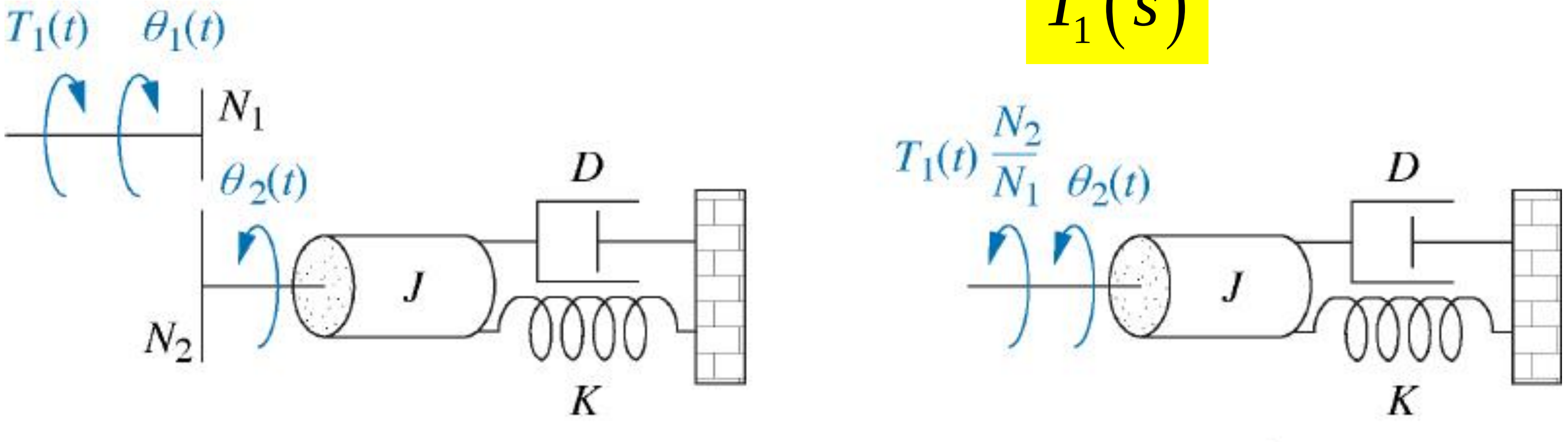
(*a*)



(*b*)

TRANSFER FUNCTIONS FOR SYSTEMS WITH GEARS

Determine transfer function $\frac{\theta_2(s)}{T_1(s)}$



(a) (b)

$$\frac{T_2}{T_1} = \frac{N_2}{N_1} \rightarrow T_2 = T_1 \frac{N_2}{N_1}$$

TRANSFER FUNCTIONS FOR SYSTEMS WITH GEARS

For J

$$\sum M = J\ddot{\theta}_2$$

$$-D\dot{\theta}_2 - K\theta_2 + T_2 = J\ddot{\theta}_2$$

$$J\ddot{\theta}_2 + D\dot{\theta}_2 + K\theta_2 = T_2$$

$$J\ddot{\theta}_2 + D\dot{\theta}_2 + K\theta_2 = T_2 = T_1\frac{N_2}{N_1}$$

Laplace transform with zero initial conditions

$$\left(Js^2 + Ds + K\right)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

TRANSFER FUNCTIONS FOR SYSTEMS WITH GEARS

Transfer function

$$\frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{Js^2 + Ds + K}$$