Frequency Response Analysis (FRA)

$$G(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$ Second-order transfer function

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$; Case $0 < \zeta < 1$

K – Gain; ω_n – Natural frequency (rad/sec); ζ - damping ratio

Harmonic Input $x(t) = Xsin\omega t$

X – Amplitude (m) ; ω- frequency (rad/sec)

Transfer function $\frac{Y(s)}{X(s)} = G(s)$

Steady-state response: Output

$$y_{ss}(t) = X|G(j\omega)|sin(\omega t + \varphi)$$

$s = j\omega; j = \sqrt{-1};$

$|G(j\omega)| -$ Magnitude of $(j\omega)$; $\varphi -$Phase of $G(j\omega)$

$$|G(j\omega)| = \sqrt{Re^2(\omega) + Im^2(\omega)}$$

$$\varphi = tan^{-1}\frac{Im(\omega)}{Re(\omega)};$$

$Re(\omega) -$Real part of $G(j\omega);$ $Im(\omega) -$ Imaginary part of $G(j\omega)$

Example

$G(s) = \frac{25}{s^2+5s+25}$; K =1; ω_n = 5 rad/sec; ζ = 0.5

$$G(j\omega) = \frac{25}{(j\omega)^2 + 5j\omega + 25}$$

$$= \frac{1 - 0.04\omega^2}{(1 - 0.04\omega^2)^2 + 0.04\omega^2} - j\frac{0.2\omega}{(1 - 0.04\omega^2)^2 + 0.04\omega^2}$$

Case I Harmonic Input $x(t) = 1 \sin 2\, t$ m

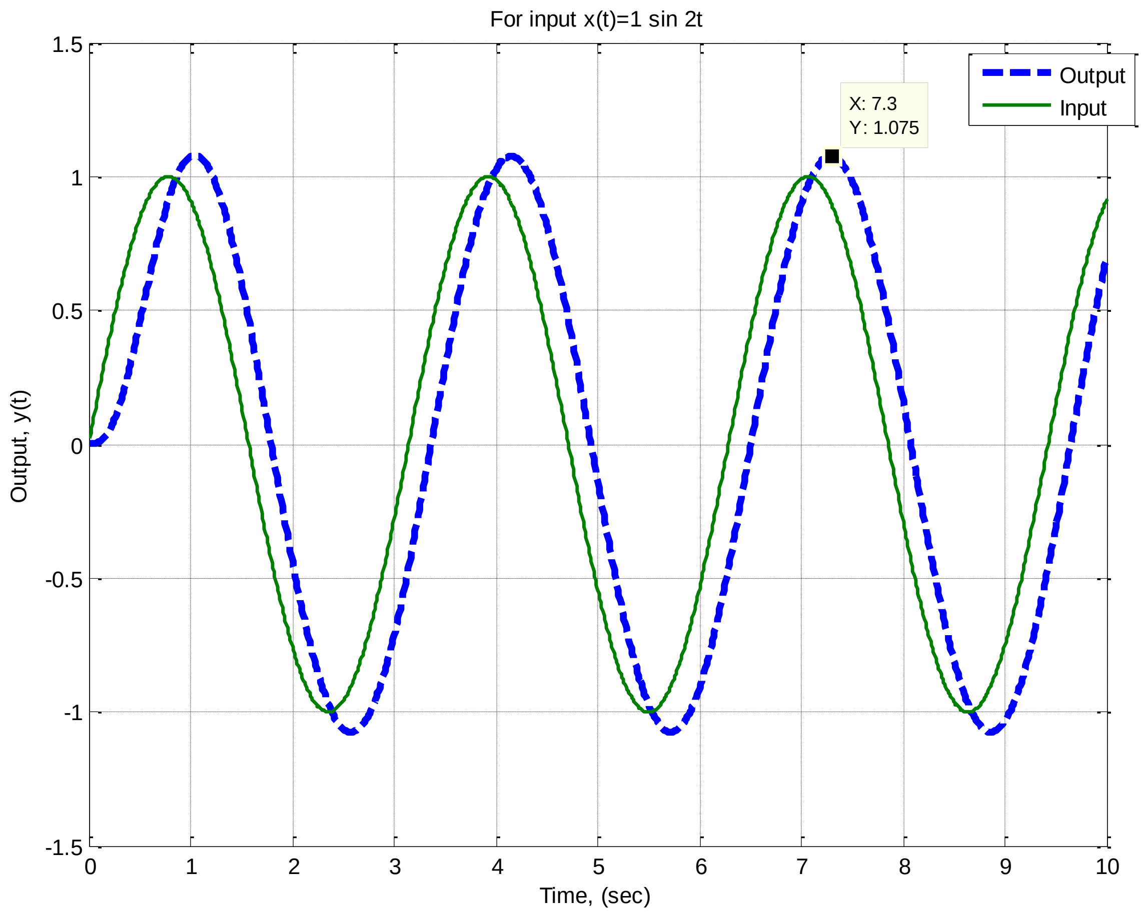
Thus, $Re(\omega) = \frac{0.84}{0.8656} 0.9704$; $Im(\omega) = -\frac{0.4}{0.8656} = -0.462$

$$|G(j\omega)| = \sqrt{(0.9704)^2 + (-0.462)^2} = 1.0747$$

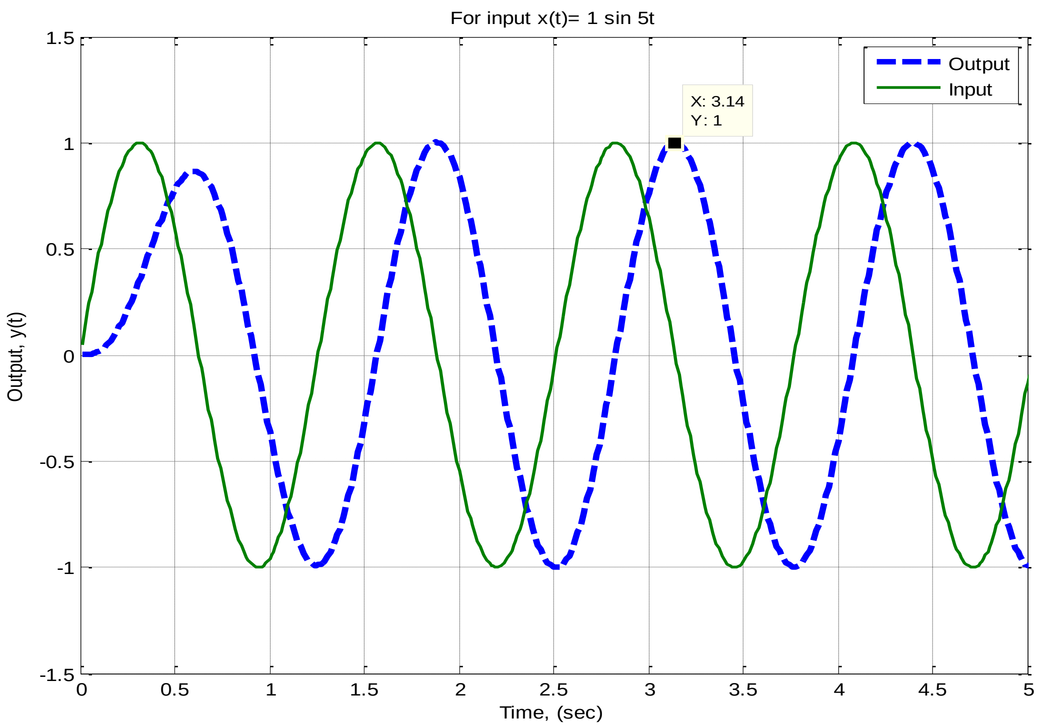
$\varphi = tan^{-1}\frac{Im(\omega)}{Re(\omega)} = tan^{-1}\frac{(-0.462)}{(0.9704)} = -25.45$ degree

Steady-state response: Output

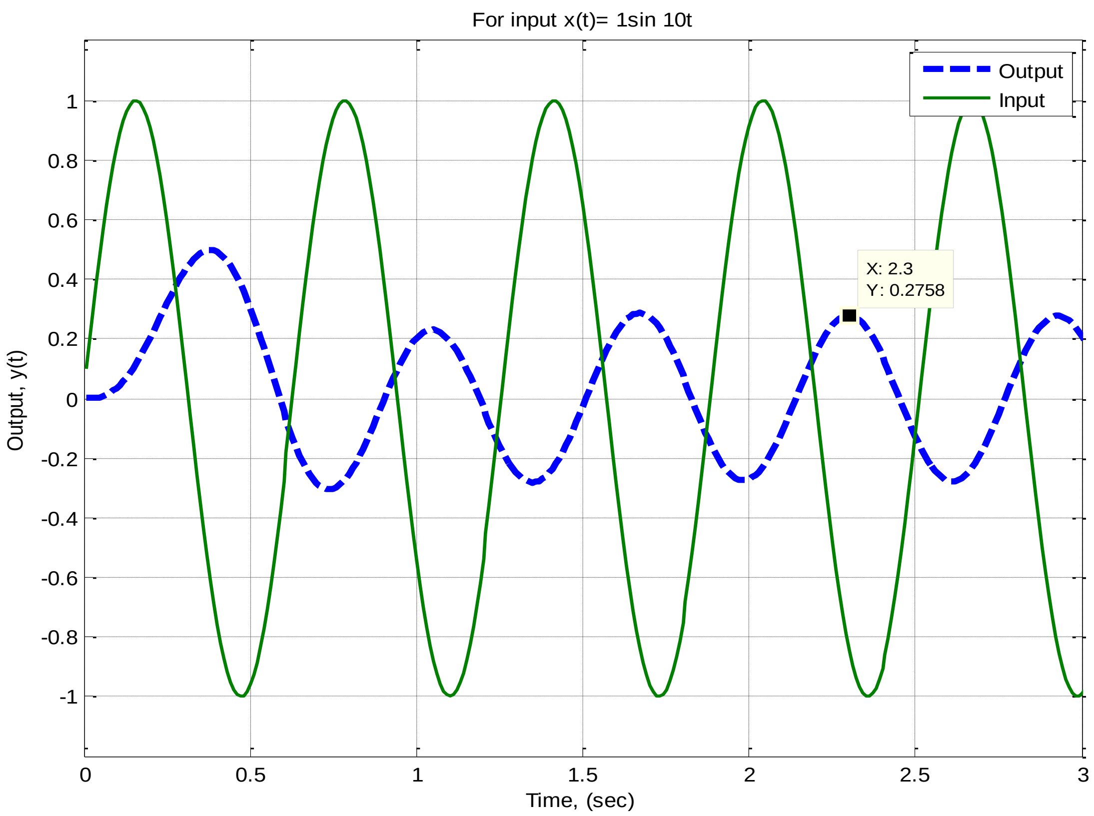
$y_{ss}(t) = X|G(j\omega)|sin(\omega t + \varphi) = 1(1.0747)sin(2t - 25.45^o)$ m

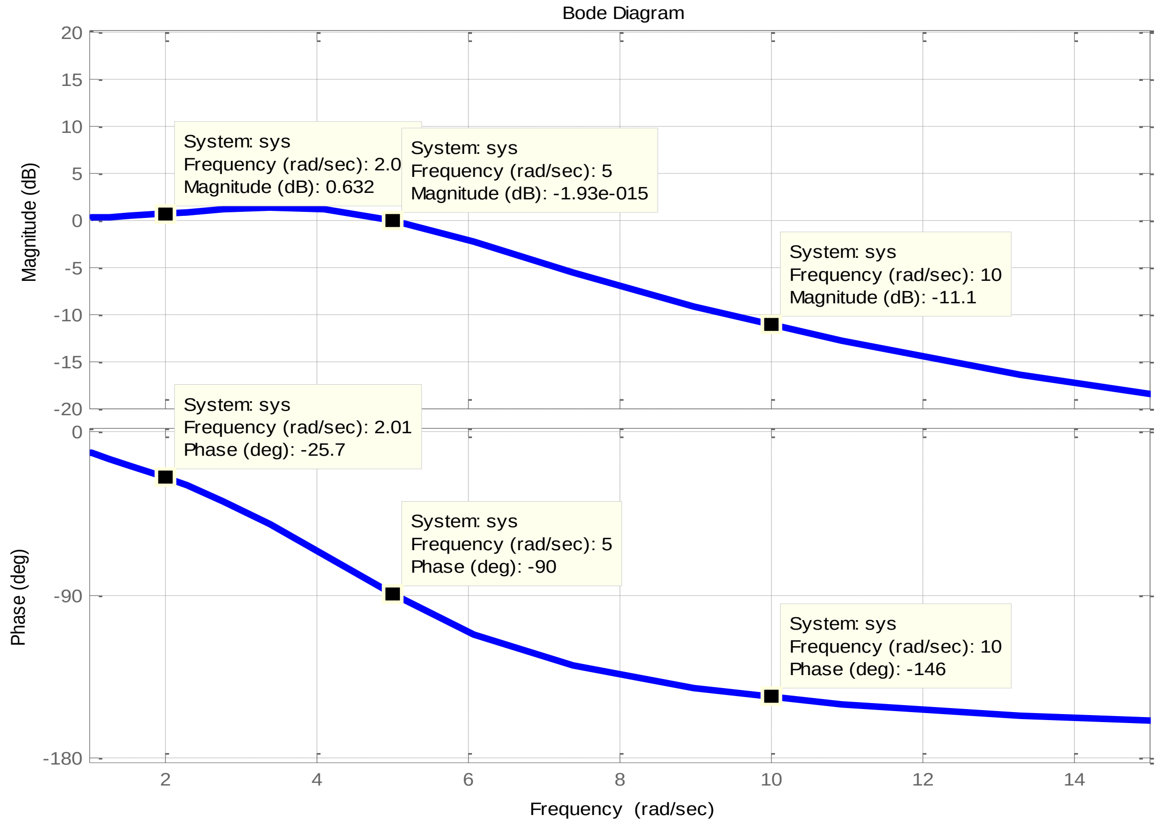


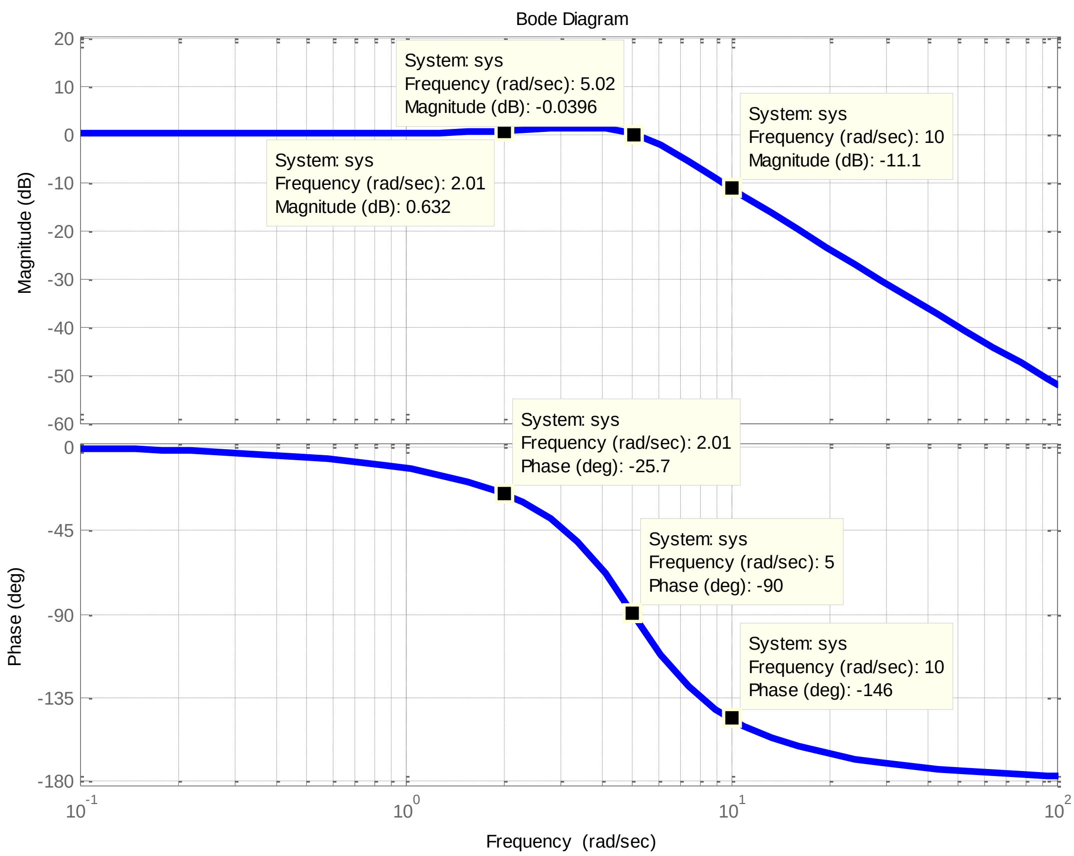
Case II Harmonic Input $x(t) = 1\sin 5\,t$ m



Case III Harmonic Input $x(t) = 1\sin 10\,t$ m




Bode Diagram
System: sys
Frequency (rad/sec): 2.0
Magnitude (dB): 0.632
System: sys
Frequency (rad/sec): 5
Magnitude (dB): -1.93e-015
System: sys
Frequency (rad/sec): 10
Magnitude (dB): -11.1
System: sys
Frequency (rad/sec): 2.01
Phase (deg): -25.7
System: sys
Frequency (rad/sec): 5
Phase (deg): -90
System: sys
Frequency (rad/sec): 10
Phase (deg): -146
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)


Bode Diagram
System: sys
Frequency (rad/sec): 5.02
Magnitude (dB): -0.0396
System: sys
Frequency (rad/sec): 10
Magnitude (dB): -11.1
System: sys
Frequency (rad/sec): 2.01
Magnitude (dB): 0.632
System: sys
Frequency (rad/sec): 2.01
Phase (deg): -25.7
System: sys
Frequency (rad/sec): 5
Phase (deg): -90
System: sys
Frequency (rad/sec): 10
Phase (deg): -146
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)