Frequency Response Analysis (FRA)

$$G(s) = \frac{K}{Ts+1}$$ First-order transfer function

K – Gain; T – Time constant

Harmonic Input $x(t) = Xsin\omega t$

X – Amplitude (m) ; ω- frequency (rad/sec)

Transfer function $\frac{Y(s)}{X(s)} = G(s)$

Steady-state response: Output

$$y_{ss}(t) = X|G(j\omega)|sin(\omega t + \varphi)$$

$s = j\omega; j = \sqrt{-1};$

$|G(j\omega)|$ – Magnitude of $(j\omega)$; φ –Phase of $G(j\omega)$

$|G(j\omega)| = \sqrt{Re^2(\omega) + Im^2(\omega)}$

$\varphi = tan^{-1}\frac{Im(\omega)}{Re(\omega)};$

$Re(\omega)$ –Real part of $G(j\omega)$; $Im(\omega)$ – Imaginary part of $G(j\omega)$

Example

$$G(s) = \frac{1}{0.2s + 1}$$

Case I Harmonic Input $x(t) = 1 \sin 2\,t$ m

Case II Harmonic Input $x(t) = 1 \sin 5\,t$ m

Case III Harmonic Input $x(t) = 1 \sin 10\,t$ m

Case I $s = j\omega$

$$G(j\omega) = \frac{1}{0.2j\omega + 1}$$

$$G(j\omega) = \frac{1}{1 + j0.2\omega} \times \frac{(1 - j0.2\omega)}{(1 - j0.2\omega)}$$

$$G(j\omega) = \frac{1 - j0.2\omega}{1 + 0.04\omega^2} = \frac{1}{1 + 0.04\omega^2} - j\frac{0.2\omega}{1 + 0.04\omega^2}$$

Thus, $Re(\omega) = \frac{1}{1+0.04\omega^2}$; $Im(\omega) = -\frac{0.2\omega}{1+0.04\omega^2}$

For case I ; input as: $x(t) = 1 \sin 2\,t$ m

X= 1 m, $\omega =$ 2 rad/sec

$$Re(\omega) = \frac{1}{1+0.04(2)^2} = 0.862;$$

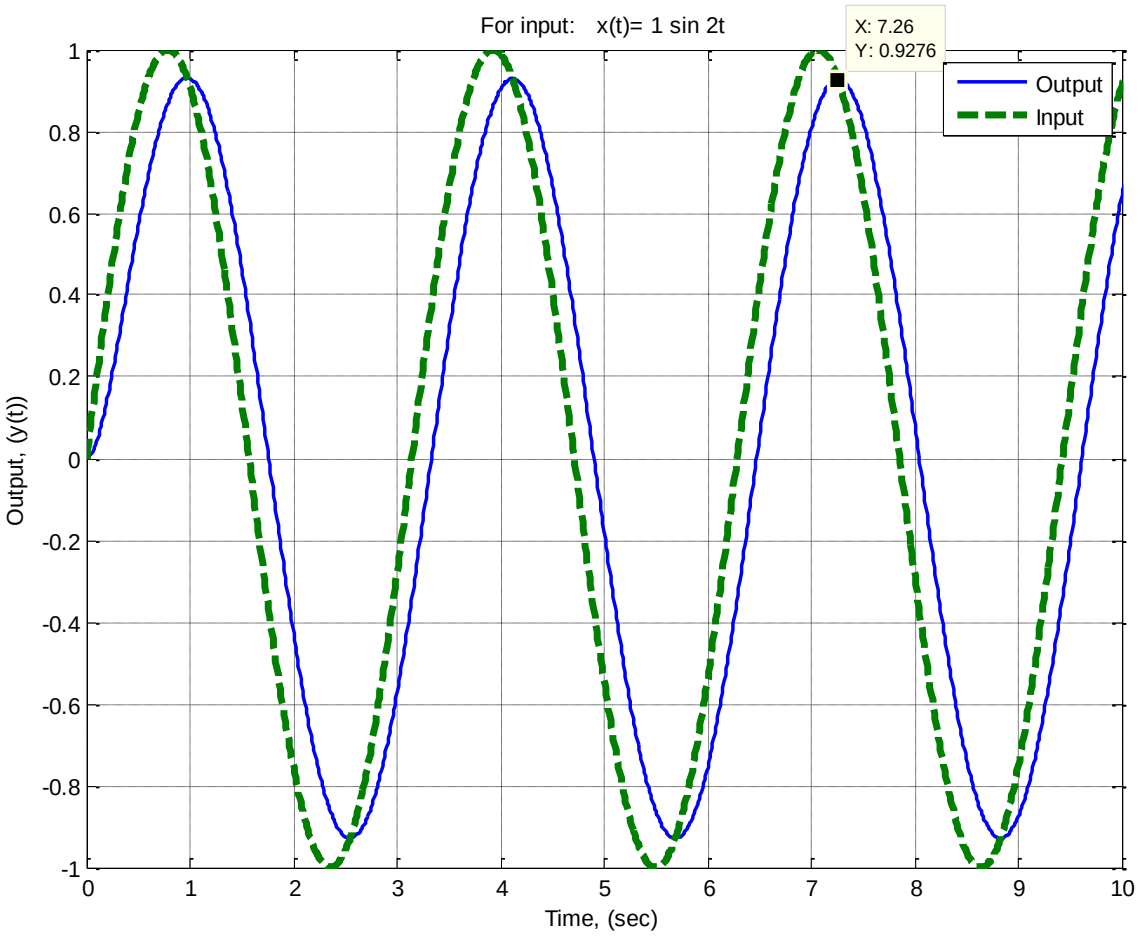
$$Im(\omega) = -\frac{0.2(2)}{1 + 0.04(2)^2} = -0.345$$

$$|G(j\omega)| = \sqrt{(0.862)^2 + (-0.345)^2} = 0.928$$

$$\varphi = tan^{-1}\frac{Im(\omega)}{Re(\omega)} = tan^{-1}\frac{(-0.1724)}{(0.862)} = -21.81 \text{ degree}$$

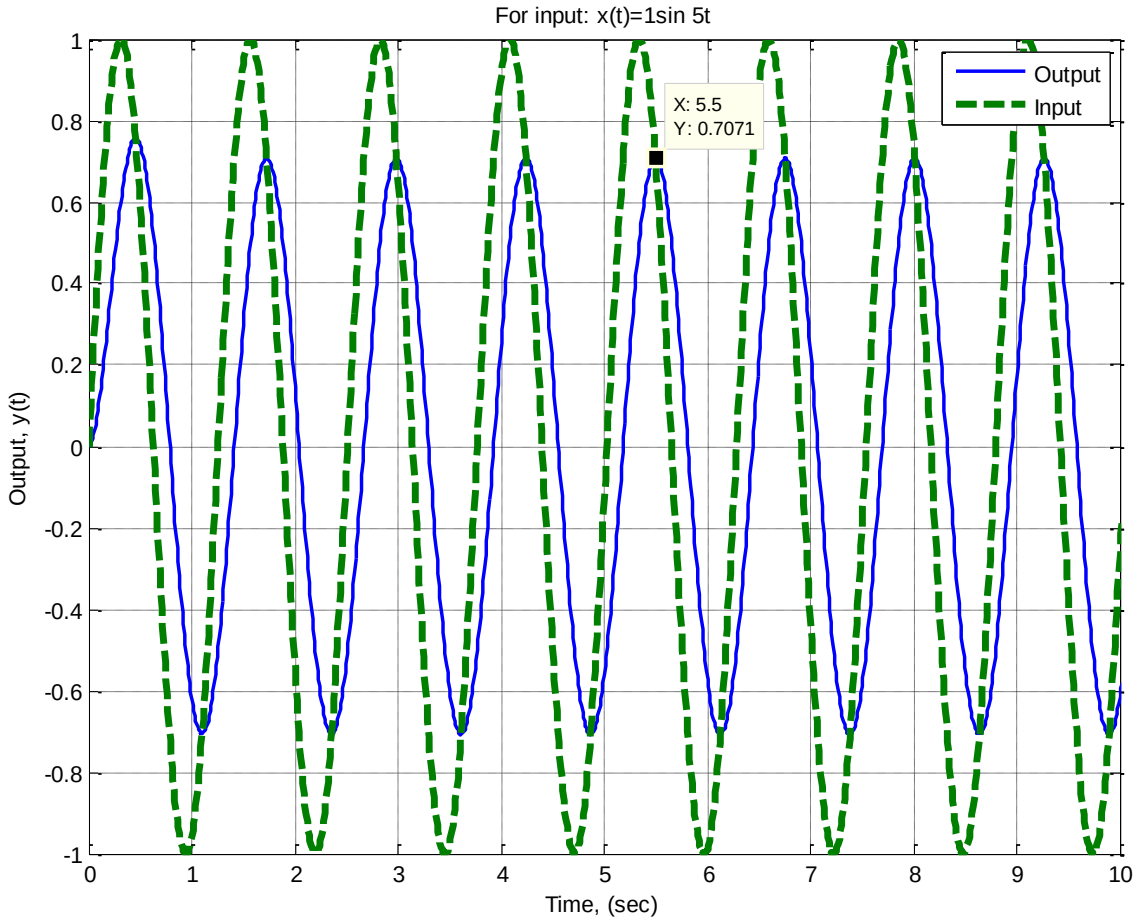
Steady-state response: Output

$$y_{ss}(t) = X|G(j\omega)|sin(\omega t + \varphi) = 1(0.928)sin(2t - 21.81^o) \text{ m}$$



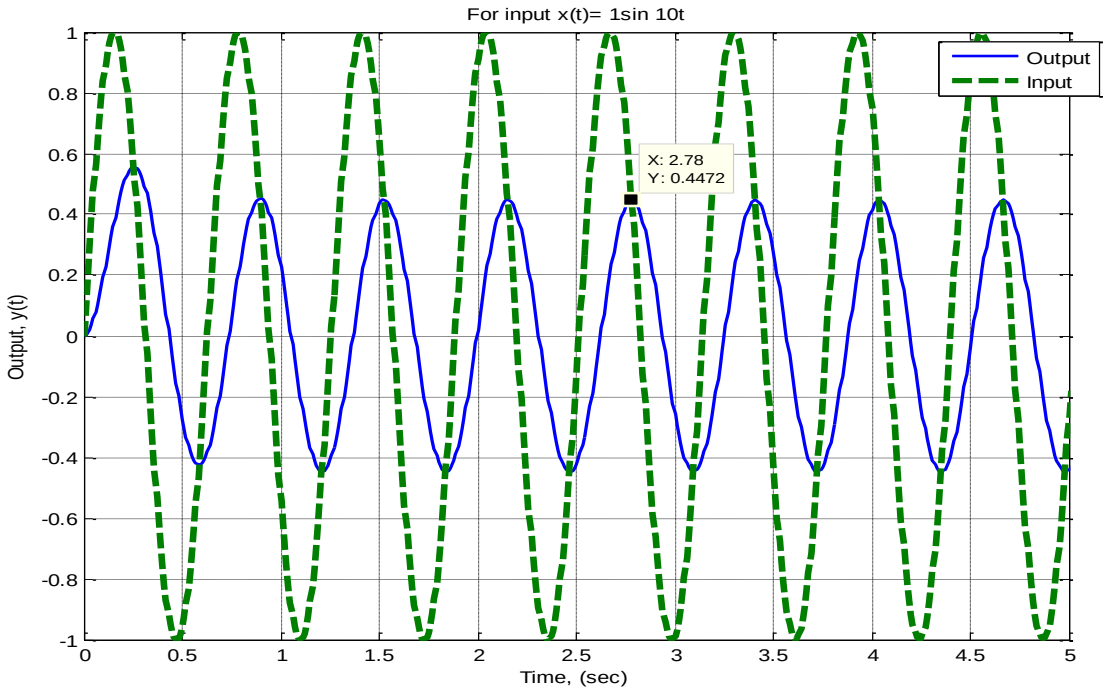
For case II ; Input $x(t) = 1 \sin 5\,t$ m

X= 1 m, $\omega = 5$ rad/sec

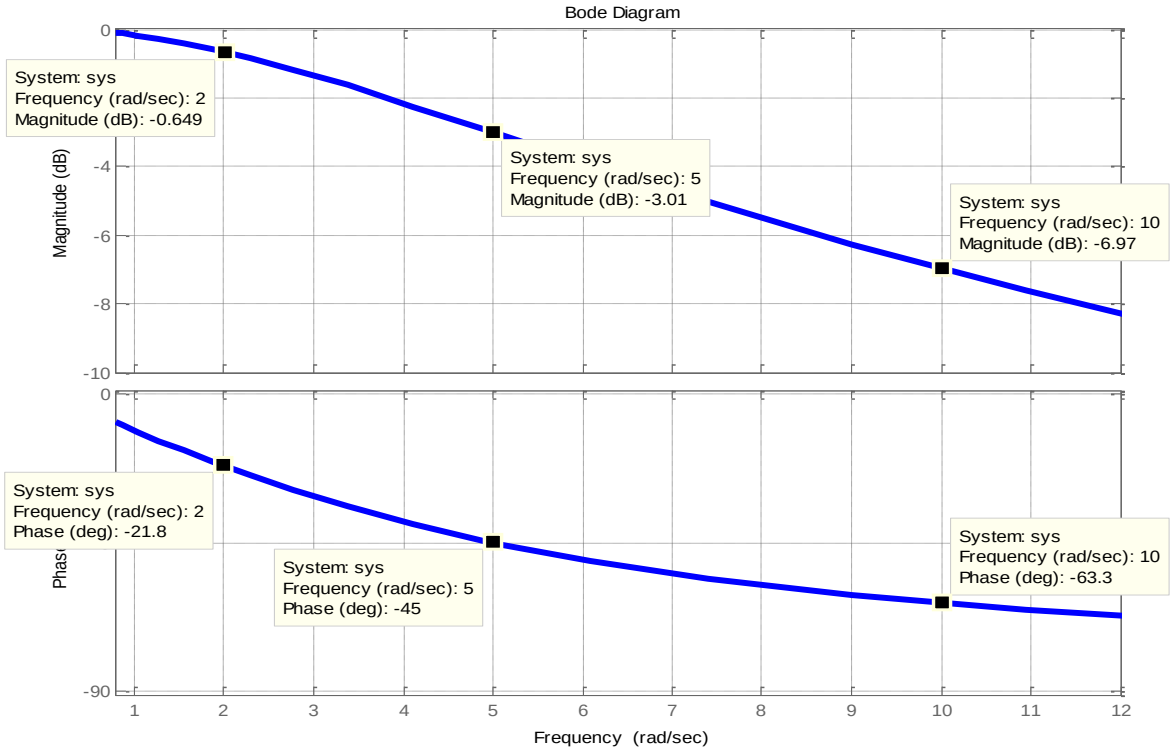


For case III ; Input $x(t) = 1 \sin 10\, t$ m

X= 1 m, $\omega = 10$ rad/sec

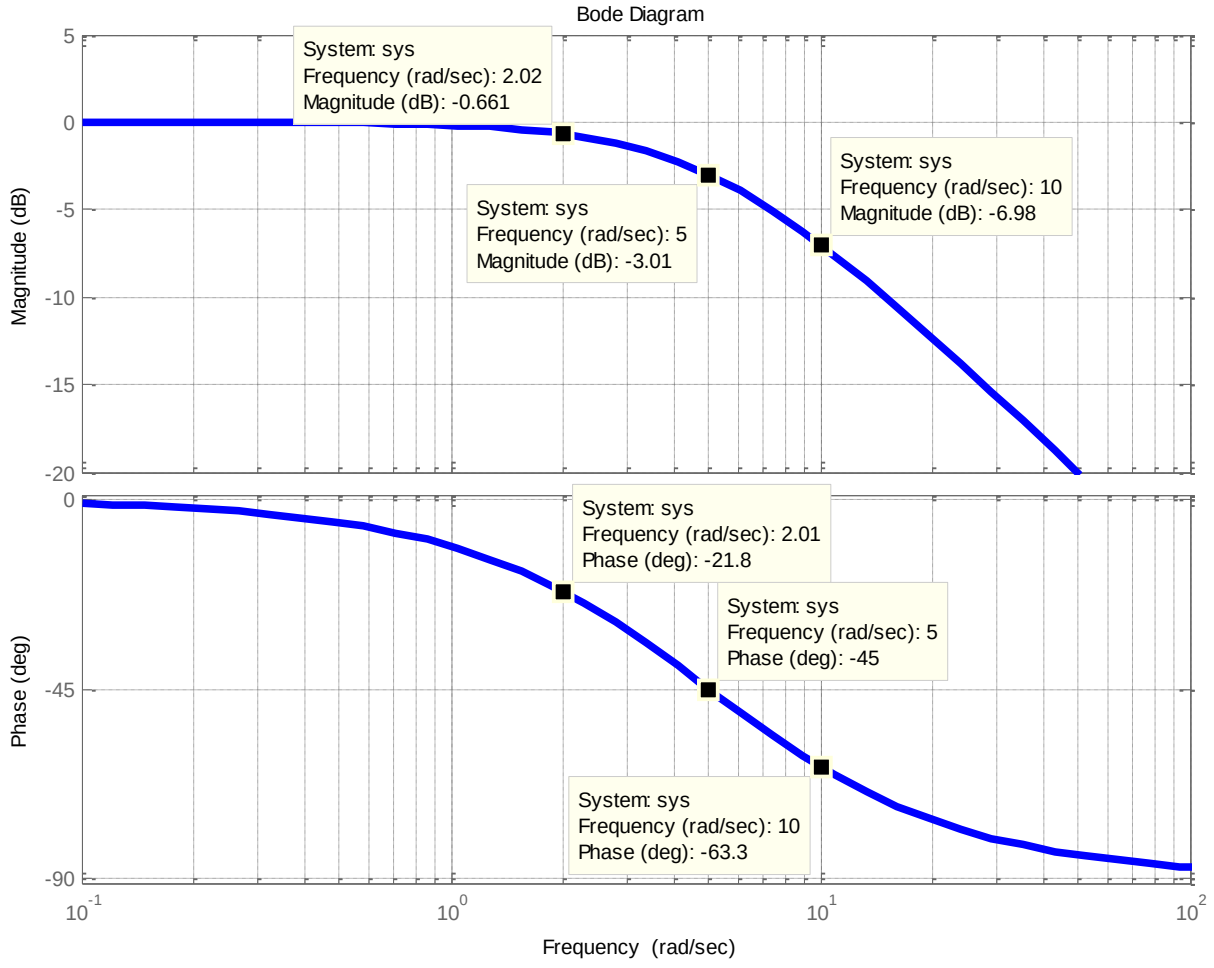


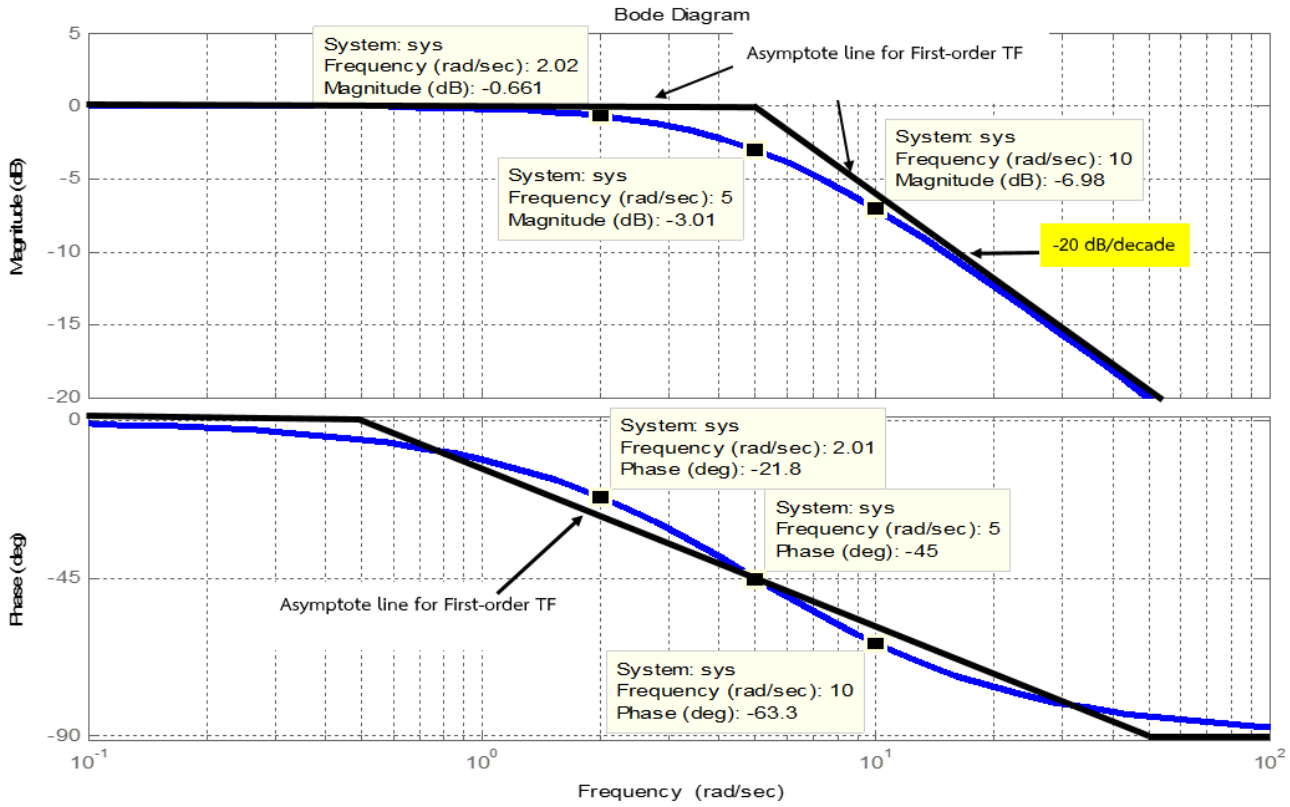
Linear scale $G(s) = \dfrac{1}{0.2s+1}$



Frequency,(rad/sec)	2	5	10
Amplitude, (m)	0.928	0.707	0.447
Amplitude, (dB) 20log(Amp)	-0.649	-3.01	-6.99
Phase, degree	-21.8	-45	-63.43

Logarithm scale $G(s) = \frac{1}{0.2s+1}$




Bode Diagram
System: sys
Frequency (rad/sec): 2.02
Magnitude (dB): -0.661
Asymptote line for First-order TF
System: sys
Frequency (rad/sec): 5
Magnitude (dB): -3.01
System: sys
Frequency (rad/sec): 10
Magnitude (dB): -6.98
-20 dB/decade
Magnitude (dB)
5
0
-5
-10
-15
-20
System: sys
Frequency (rad/sec): 2.01
Phase (deg): -21.8
System: sys
Frequency (rad/sec): 5
Phase (deg): -45
Asymptote line for First-order TF
System: sys
Frequency (rad/sec): 10
Phase (deg): -63.3
Phase (deg)
0
-45
-90
10^-1
10^0
10^1
10^2
Frequency (rad/sec)