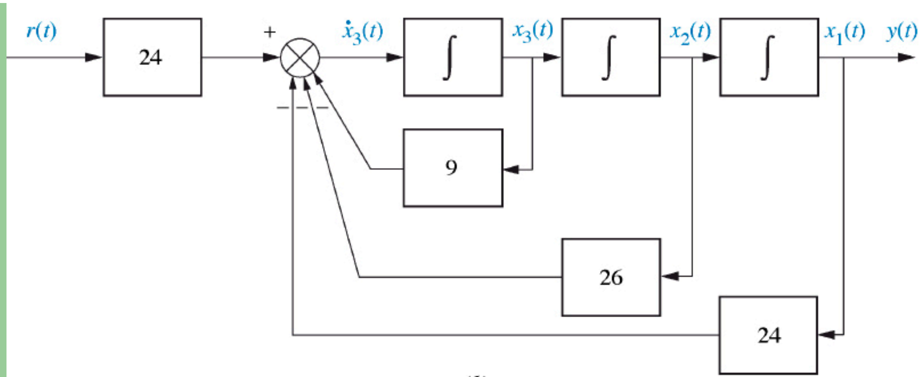
Modeling in the Time Domain



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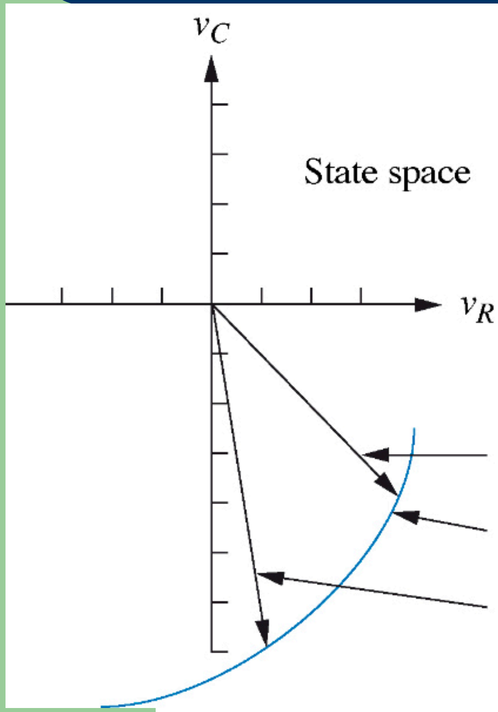
System & Control Engineering Lab.
School of Mechanical Engineering

CHAPTER OBJECTIVES

- How to find a mathematical model, called a state space representation, for a linear, time-invariant system
- How to convert between transfer function and state-space models
- How to linearize a state-space representation

THE GENERAL STATE-SPACE REPRESENTATION



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$\mathbf{x}$ = state vector

$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time

$\mathbf{y}$ = output vector

$\mathbf{u}$ = input or control vector

$\mathbf{A}$ = system matrix

$\mathbf{B}$ = input matrix

$\mathbf{C}$ = output matrix

$\mathbf{D}$ = feedforward matrix

State-space equation

Linearized state equation and output equation

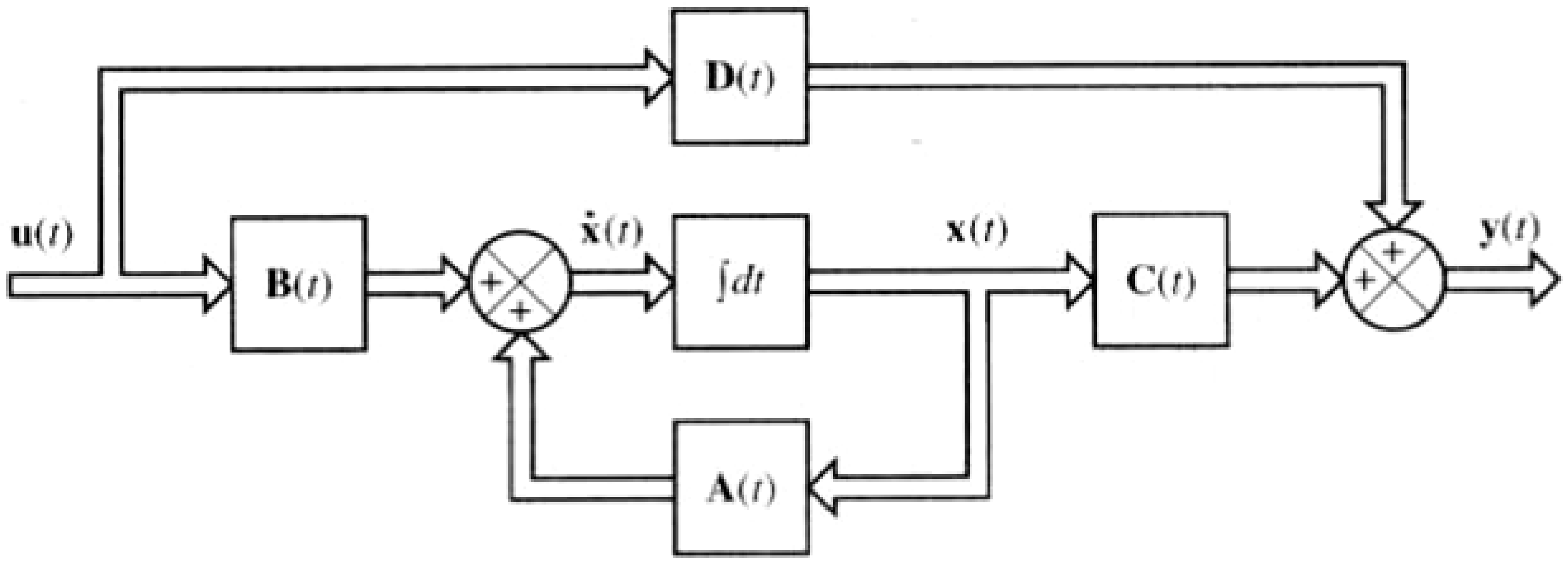
$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

A(t) – State matrix; B(t) – Input matrix

C(t) – Output matrix; D(t) – Direct transmission matrix

State-space equation



Block diagram the linear, Continuous time control system represented in state space

THE GENERAL STATE-SPACE REPRESENTATION

As an example, for a linear, time-invariant, second-order system with a single input $v(t)$, the state equations could take on the following form:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1v(t) \tag{3.20a}$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2v(t) \tag{3.20b}$$

where x_1 and x_2 are the state variables. If there is a single output, the output equation could take on the following form:

$$y = c_1x_1 + c_2x_2 + d_1v(t) \tag{3.21}$$

State-space equation

$$\dot{x}_1(t) = f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots u_r; t)$$
$$\dot{x}_2(t) = f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots u_r; t)$$
$$\vdots$$
$$\dot{x}_n(t) = f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots u_r; t)$$

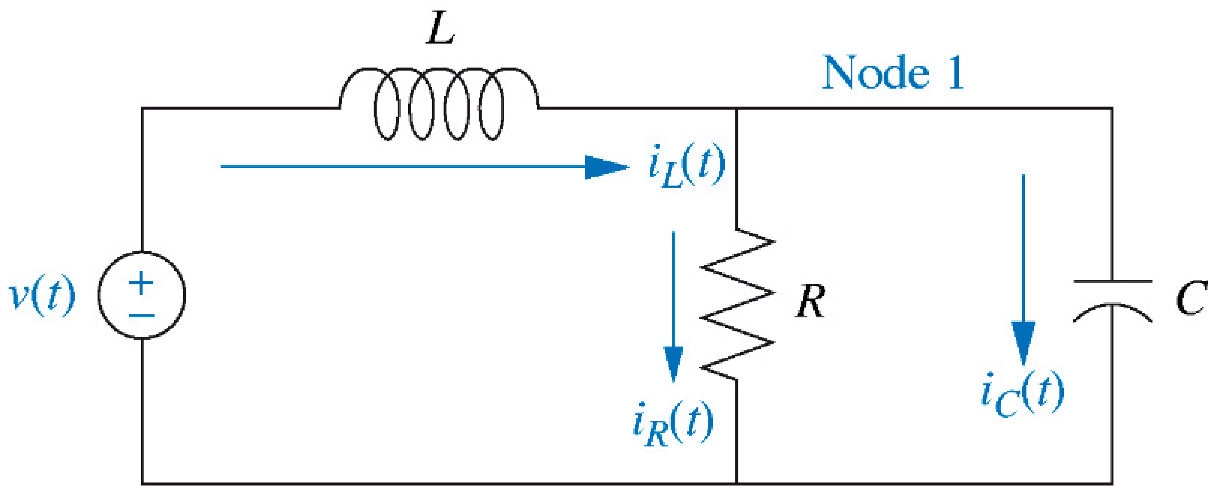
$$\dot{x}(t) = f(x, u, t)$$

State-space equation

$$y_1(t) = g_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots u_r; t)$$
$$y_2(t) = g_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots u_r; t)$$
$$\vdots$$
$$y_m(t) = g_m(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots u_r; t)$$

$$\dot{y}(t) = g(x, u, t)$$

THE GENERAL STATE-SPACE REPRESENTATION



$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L$$

$$\frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}v(t)$$

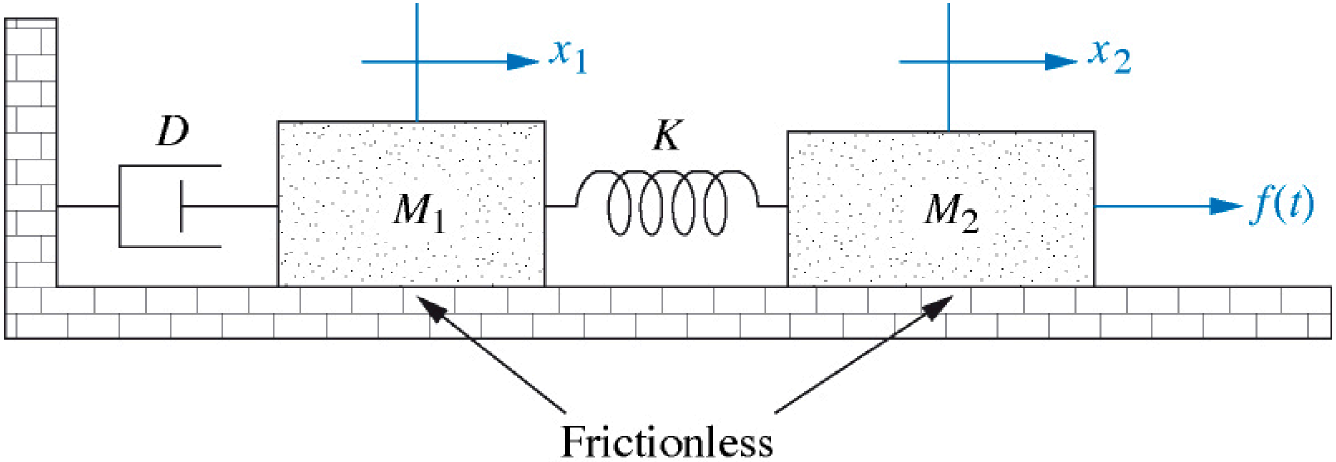
$$i_R = \frac{1}{R}v_C$$

THE GENERAL STATE-SPACE REPRESENTATION

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t)$$

$$i_R = \begin{bmatrix} 1/R & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

THE GENERAL STATE-SPACE REPRESENTATION



$$M_1 \frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$

$$-Kx_1 + M_2\frac{d^2x_2}{dt^2} + Kx_2 = f(t)$$

THE GENERAL STATE-SPACE REPRESENTATION

$$\frac{dx_1}{dt} = \qquad\qquad + v_1$$

$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2$$

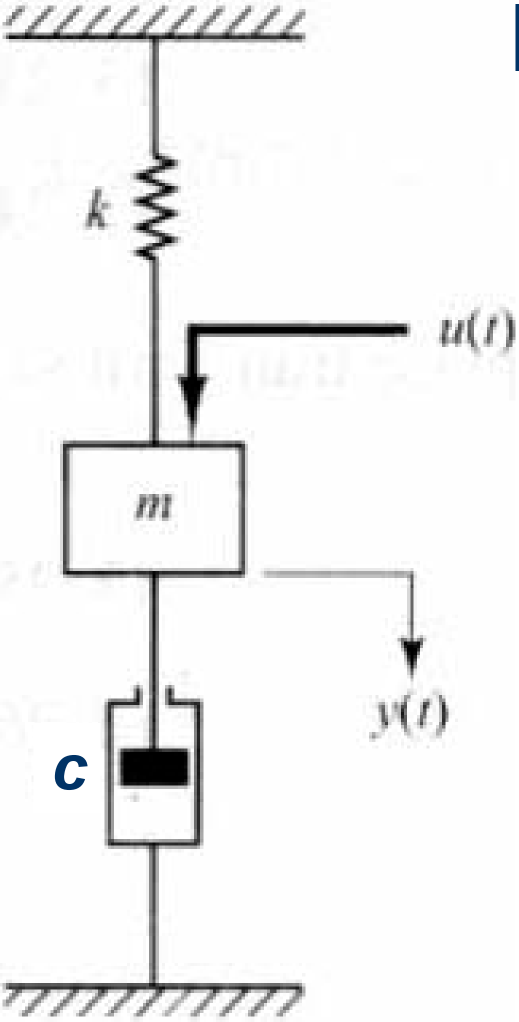
$$\frac{dx_2}{dt} = \qquad\qquad + v_2$$

$$\frac{dv_2}{dt} = +\frac{K}{M_2}x_1 \qquad - \frac{K}{M_2}x_2 \qquad + \frac{1}{M_2}f(t)$$

THE GENERAL STATE-SPACE REPRESENTATION

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$

State-space equation
Ex: Mechanical system



$$u(t) - c\dot{y} - ky = m\ddot{y}$$

$$u(t) = m\ddot{y} + c\dot{y} + ky$$

State-space equation
Ex: Mechanical system

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}\left(-ky - c\dot{y}\right) + \frac{1}{m}u = -\frac{k}{m}y - \frac{c}{m}\dot{y} + \frac{1}{m}u$$

State-space equation
Ex: Mechanical system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

State-space equation
Ex: Mechanical system

Output

$$y = x_1$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-space equation
Ex: Mechanical system

Standard form

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

CONVERTING A TRANSFER FUNCTION TO STATE SPACE

- phase variables

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u \qquad (3.48)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n$$

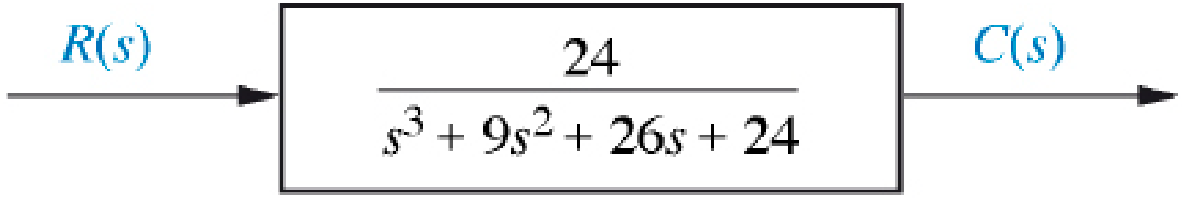
$$\dot{x}_n = -a_0 x_1 - a_1 x_2 \cdots - a_{n-1} x_n + b_0 u$$

CONVERTING A TRANSFER FUNCTION TO STATE SPACE

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u \tag{3.52}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

CONVERTING A TRANSFER FUNCTION TO STATE SPACE



$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

$$x_1 = c$$

$$x_2 = \dot{c}$$

$$x_3 = \ddot{c}$$

CONVERTING A TRANSFER FUNCTION TO STATE SPACE

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

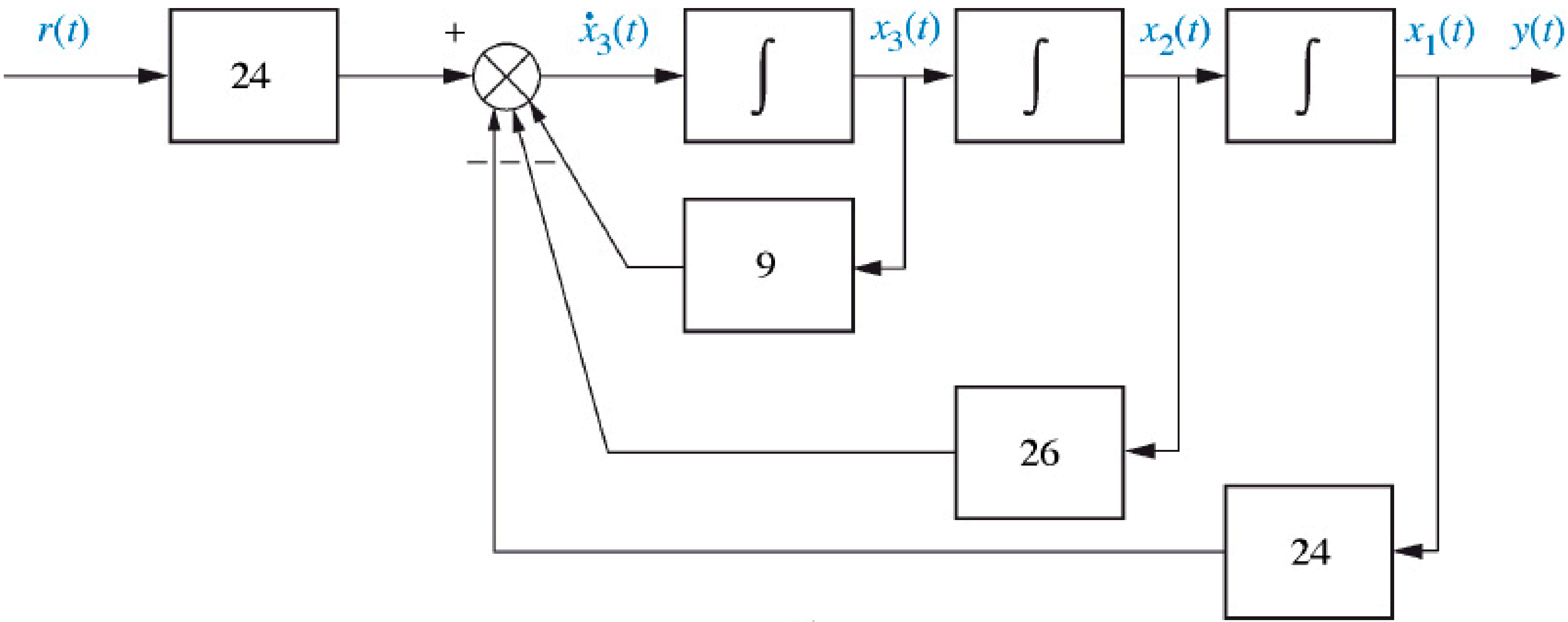
$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

$$y = c = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

CONVERTING A TRANSFER FUNCTION TO STATE SPACE



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CONVERTING FROM STATE SPACE TO A TRANSFER FUNCTION

From state-space equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Laplace Transform

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

CONVERTING FROM STATE SPACE TO A TRANSFER FUNCTION

$$sX(s) - AX(s) = BU(s)$$

$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1}BU(s)$$

CONVERTING FROM STATE SPACE TO A TRANSFER FUNCTION

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$Y(s) = \left[C(sI - A)^{-1}B + D\right]U(s)$$

CONVERTING FROM STATE SPACE TO A TRANSFER FUNCTION

Transfer function

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D = G(s)$$

CONVERTING FROM STATE SPACE TO A TRANSFER FUNCTION: EX

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

CONVERTING FROM STATE SPACE TO A TRANSFER FUNCTION: EX

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D = G(s)$$

$$T(s) = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$