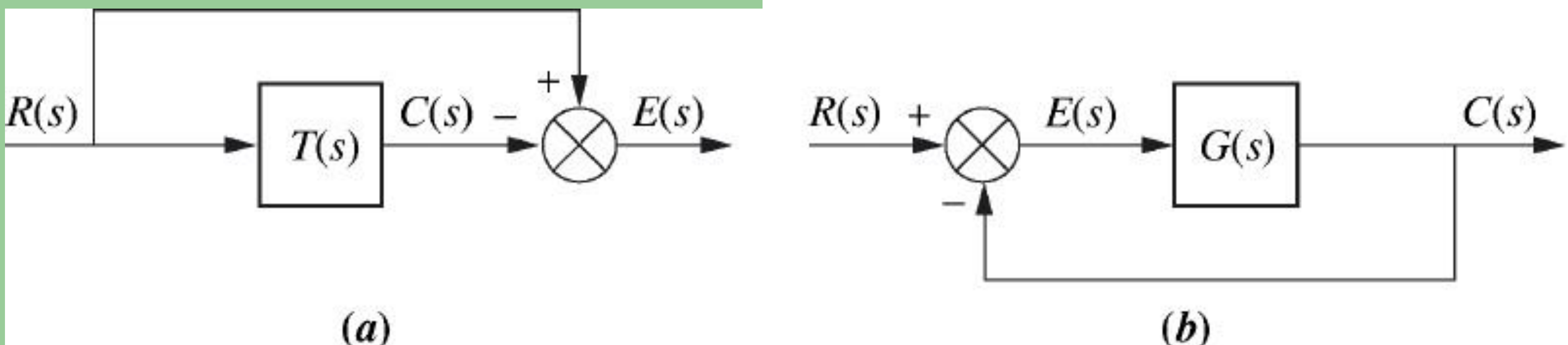
Steady-State Errors



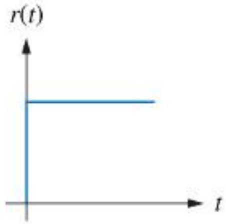
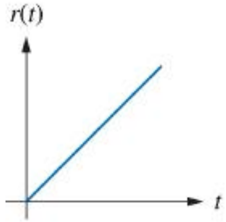
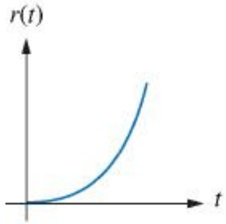
System & Control Engineering Lab.
School of Mechanical Engineering

CHAPTER OBJECTIVES

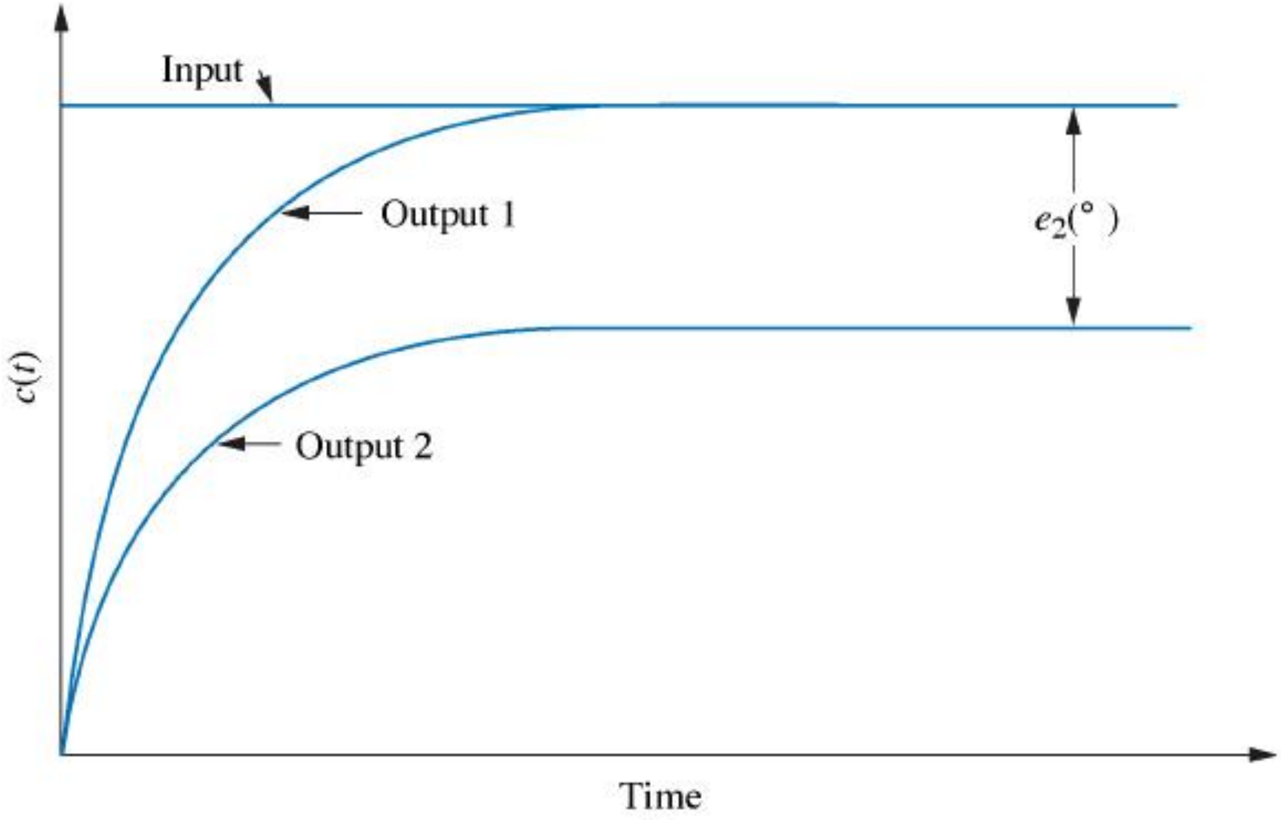
- How to find the steady-state error for a unity feedback system
- How to specify a system’s steady-state error performance
- How to find the steady-state error for disturbance inputs
- How to find the steady-state error for nonunity feedback systems

DEFINITION AND TEST INPUTS

TABLE 7.1 Test waveforms for evaluating steady-state errors of position control systems

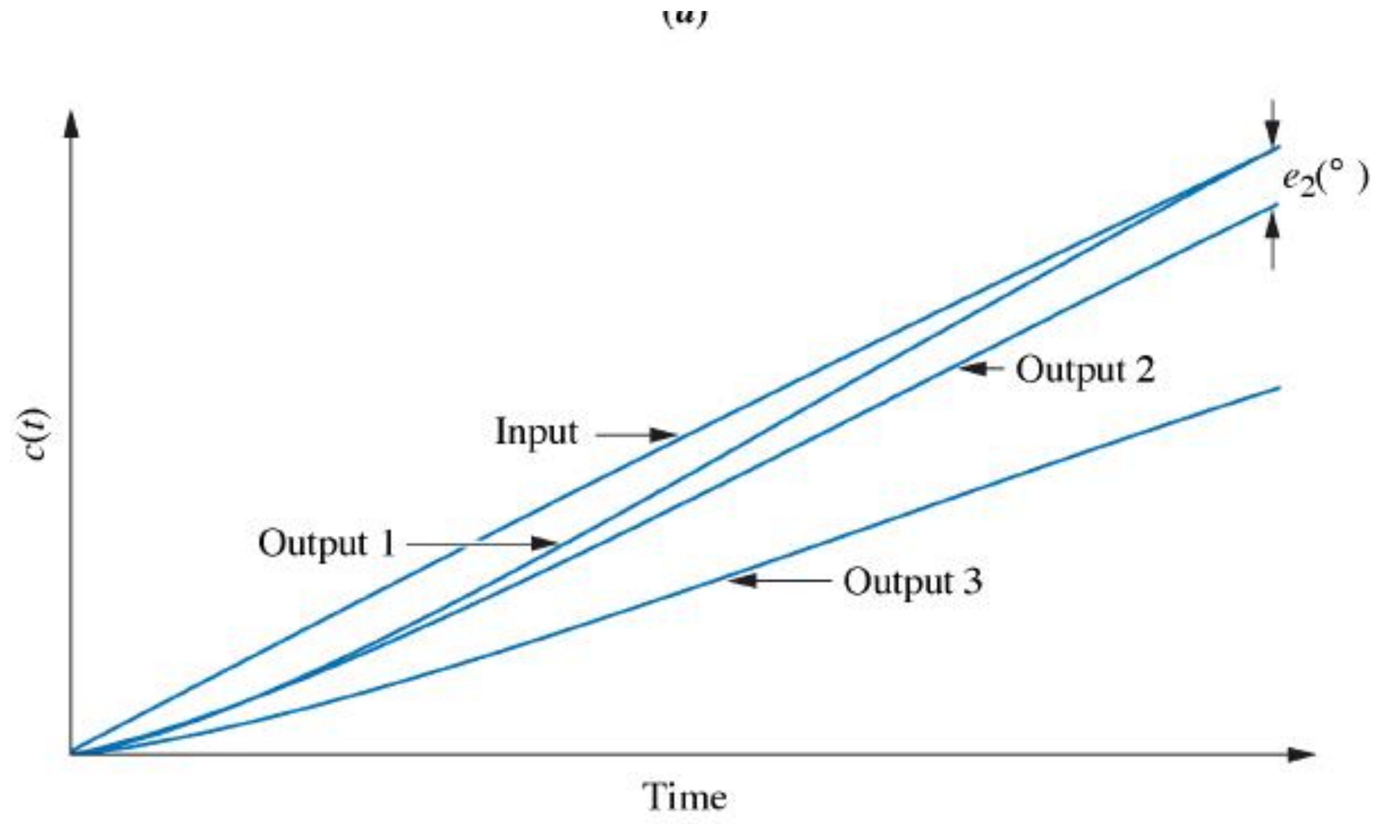
Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Evaluating Steady-State Errors



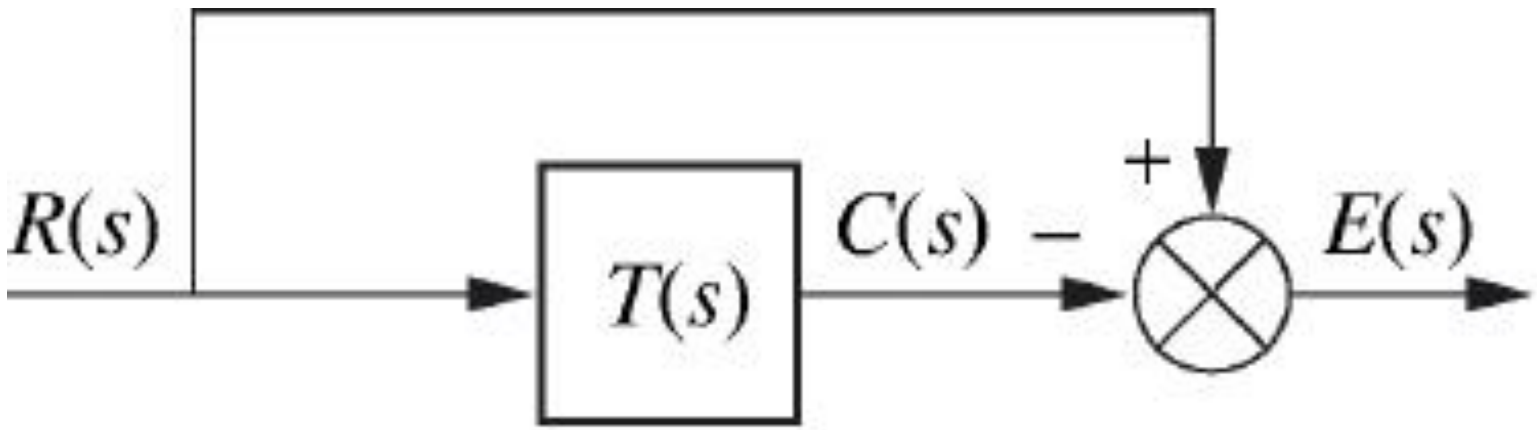
(*a*)

Evaluating Steady-State Errors



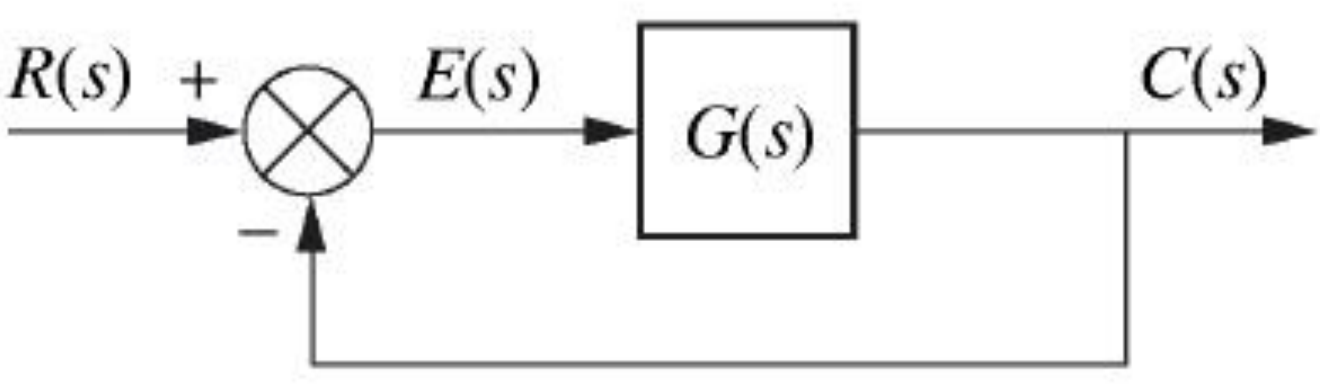
(*b*)

Sources of Steady-State Error



(a)

Sources of Steady-State Error



(b)

Unity feedback system

$$E(s) = R(s) - C(s)$$

Steady-State Error for Unity Feedback Systems

Final value theorem

$$e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

Steady-State Errors

[redacted]

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

The transfer function between the error signal and input signal is

$$E(s) = \frac{1}{1+G(s)} R(s)$$

Steady-State Error for Unity Feedback Systems

The steady-state error is

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Steady-State Error for Unity Feedback Systems: Step Input

$$e(\infty) = e_{\text{step}}(\infty) = \lim_{s \to 0} \frac{s(1/s)}{1 + G(s)} = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

$$\lim_{s \to 0} G(s)$$

Steady-State Error for Unity Feedback Systems: Ramp Input

$$e(\infty) = e_{\text{ramp}}(\infty) = \lim_{s\to 0} \frac{s(1/s^2)}{1 + G(s)} = \lim_{s\to 0} \frac{1}{s + sG(s)} = \frac{1}{\lim\limits_{s\to 0} sG(s)}$$

Steady-State Error for Unity Feedback Systems: Parabolic Input

$$e(\infty) = e_{\text{parabola}}(\infty) = \lim_{s\to 0} \frac{s(1/s^3)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2 + s^2G(s)} = \frac{1}{\lim_{s\to 0} s^2G(s)}$$

Static Error Constants

In the previous section we derived the following relationships for steady-state error. For a step input, $u(t)$,

$$e(\infty) = e_{\text{step}}(\infty) = \frac{1}{1 + \lim_{s \to 0} G(s)} \tag{7.30}$$

For a ramp input, $tu(t)$,

$$e(\infty) = e_{\text{ramp}}(\infty) = \frac{1}{\lim_{s \to 0} sG(s)} \tag{7.31}$$

For a parabolic input, $\frac{1}{2}t^2u(t)$,

$$e(\infty) = e_{\text{parabola}}(\infty) = \frac{1}{\lim_{s \to 0} s^2G(s)} \tag{7.32}$$

Static Error Constants

position constant, K_p, where

$$K_p = \lim_{s \to 0} G(s) \tag{7.33}$$

velocity constant, K_v, where

$$K_v = \lim_{s \to 0} sG(s) \tag{7.34}$$

acceleration constant, K_a, where

$$K_a = \lim_{s \to 0} s^2 G(s) \tag{7.35}$$

Classification of Control Systems

Open-loop transfer function

$$G(s)=\frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

Type 0,1,2…., if N=0,1,2……

Classification of Control Systems

$$G(s) = \frac{5}{s^2 + 4s + 3} = \frac{5}{(s+1)(s+3)} \rightarrow \text{type } 0$$

$$G(s) = \frac{1}{s(3s+1)} \rightarrow \text{type } 1$$

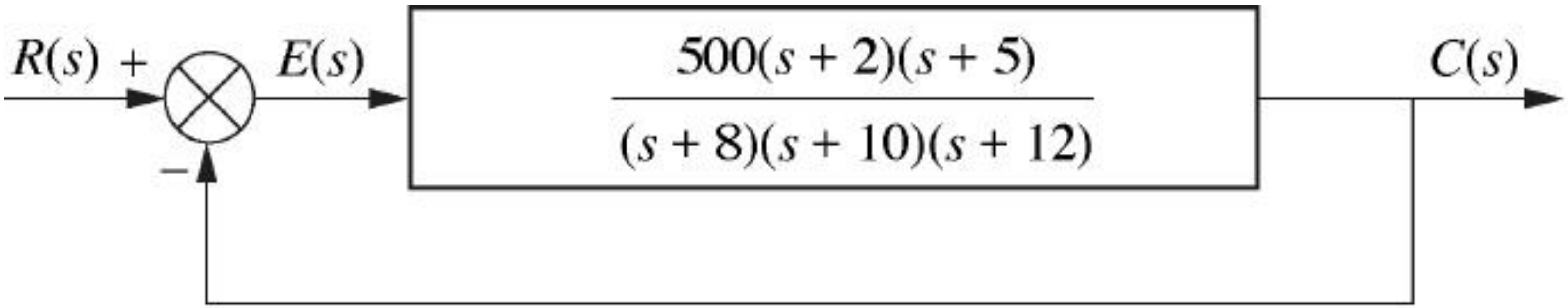
$$G(s) = \frac{5}{s^2(s+3)} \rightarrow \text{type } 2$$

System Type

TABLE 7.2 Relationships between input, system type, static error constants, and steady-state errors

		Type 0		Type 1		Type 2	
Input	Steady-state error formula	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

Steady-State Error for Unity Feedback Systems: Ex



(a)

Steady-State Error for Unity Feedback Systems: Ex

SOLUTION: First verify that all closed-loop systems shown are indeed stable. For this example we leave out the details. Next, for Figure 7.7(a),

$$K_p = \lim_{s \to 0} G(s) = \frac{500 \times 2 \times 5}{8 \times 10 \times 12} = 5.208 \tag{7.36}$$

$$K_v = \lim_{s \to 0} sG(s) = 0 \tag{7.37}$$

$$K_a = \lim_{s \to 0} s^2 G(s) = 0 \tag{7.38}$$

Steady-State Error for Unity Feedback Systems: Ex

Thus, for a step input,

$$e(\infty) = \frac{1}{1 + K_p} = 0.161$$

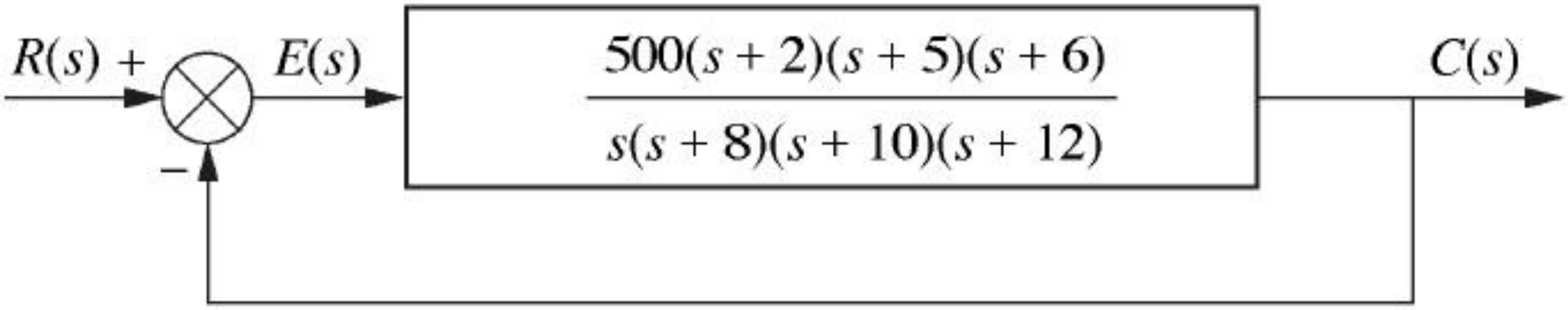
For a ramp input,

$$e(\infty) = \frac{1}{K_v} = \infty$$

For a parabolic input,

$$e(\infty) = \frac{1}{K_a} = \infty$$

Steady-State Error for Unity Feedback Systems: Ex



(b)

Steady-State Error for Unity Feedback Systems: Ex

Now, for Figure 7.7(b),

$$K_p = \lim_{s \to 0} G(s) = \infty$$

$$K_v = \lim_{s \to 0} sG(s) = \frac{500 \times 2 \times 5 \times 6}{8 \times 10 \times 12} = 31.25$$

$$K_a = \lim_{s \to 0} s^2 G(s) = 0$$

Steady-State Error for Unity Feedback Systems: Ex

Thus, for a step input,

$$e(\infty) = \frac{1}{1 + K_p} = 0$$

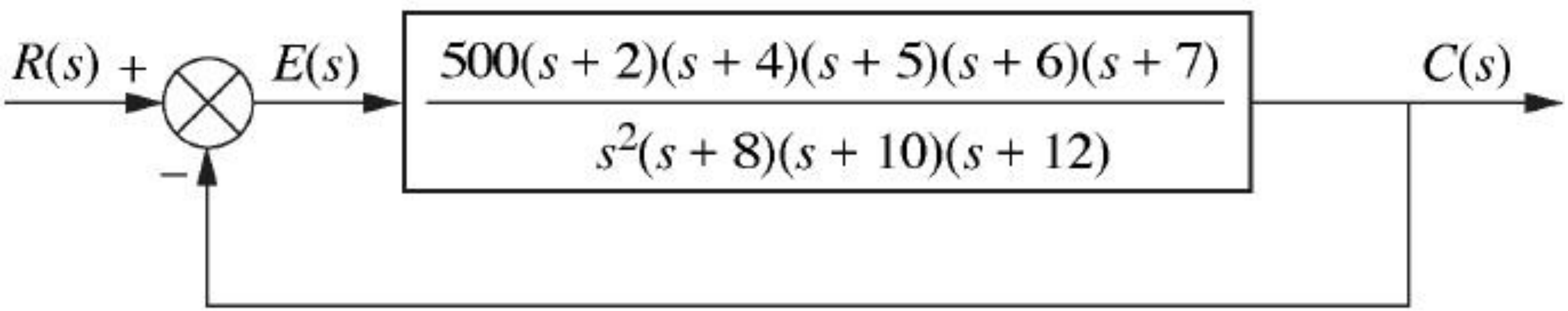
For a ramp input,

$$e(\infty) = \frac{1}{K_v} = \frac{1}{31.25} = 0.032$$

For a parabolic input,

$$e(\infty) = \frac{1}{K_a} = \infty$$

Steady-State Error for Unity Feedback Systems: Ex



(c)

Steady-State Error for Unity Feedback Systems: Ex

$$K_p = \lim_{s \to 0} G(s) = \infty$$

$$K_v = \lim_{s \to 0} sG(s) = \infty$$

$$K_a = \lim_{s \to 0} s^2 G(s) = \frac{500 \times 2 \times 4 \times 5 \times 6 \times 7}{8 \times 10 \times 12} = 875$$

Steady-State Error for Unity Feedback Systems: Ex

Thus, for a step input,

$$e(\infty) = \frac{1}{1 + K_p} = 0$$

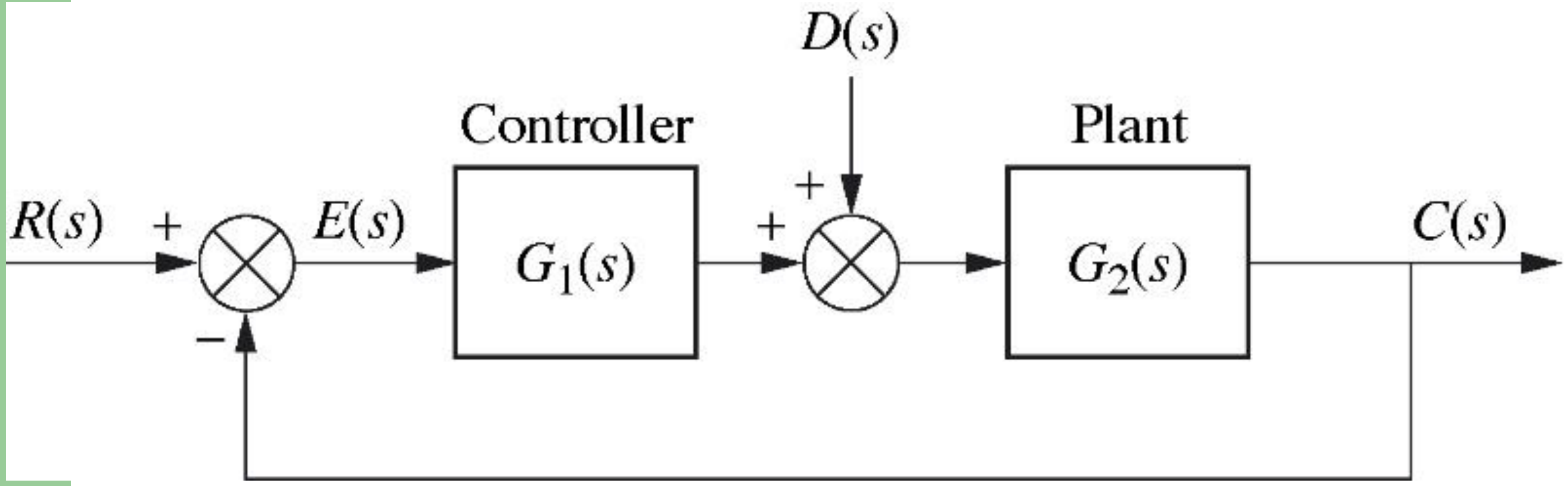
For a ramp input,

$$e(\infty) = \frac{1}{K_v} = 0$$

For a parabolic input,

$$e(\infty) = \frac{1}{K_a} = \frac{1}{875} = 1.14 \times 10^{-3}$$

Steady-State Error for Disturbances



$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$$

Steady-State Error for Disturbances

But

$$C(s) = R(s) - E(s)$$

Substituting Eq. (7.59) into Eq. (7.58) and solving for $E(s)$, we obtain

$$E(s) = \frac{1}{1 + G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)}D(s)$$

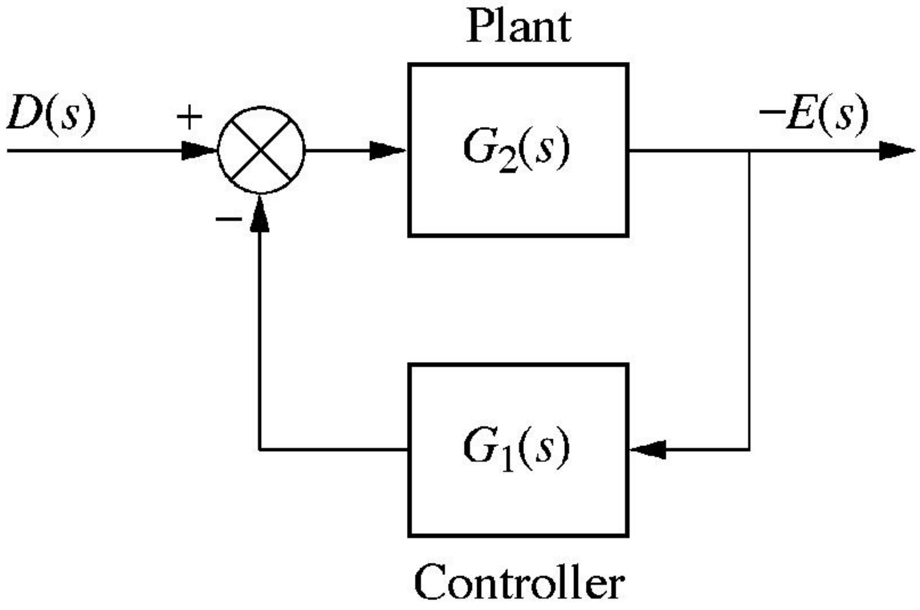
$$\begin{aligned} e(\infty) &= \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1 + G_1(s)G_2(s)}R(s) - \lim_{s\to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)}D(s) \\ &= e_R(\infty) + e_D(\infty) \end{aligned}$$

Steady-State Error for Disturbances

$$e_R(\infty) = \lim_{s \to 0} \frac{s}{1 + G_1(s)G_2(s)} R(s)$$

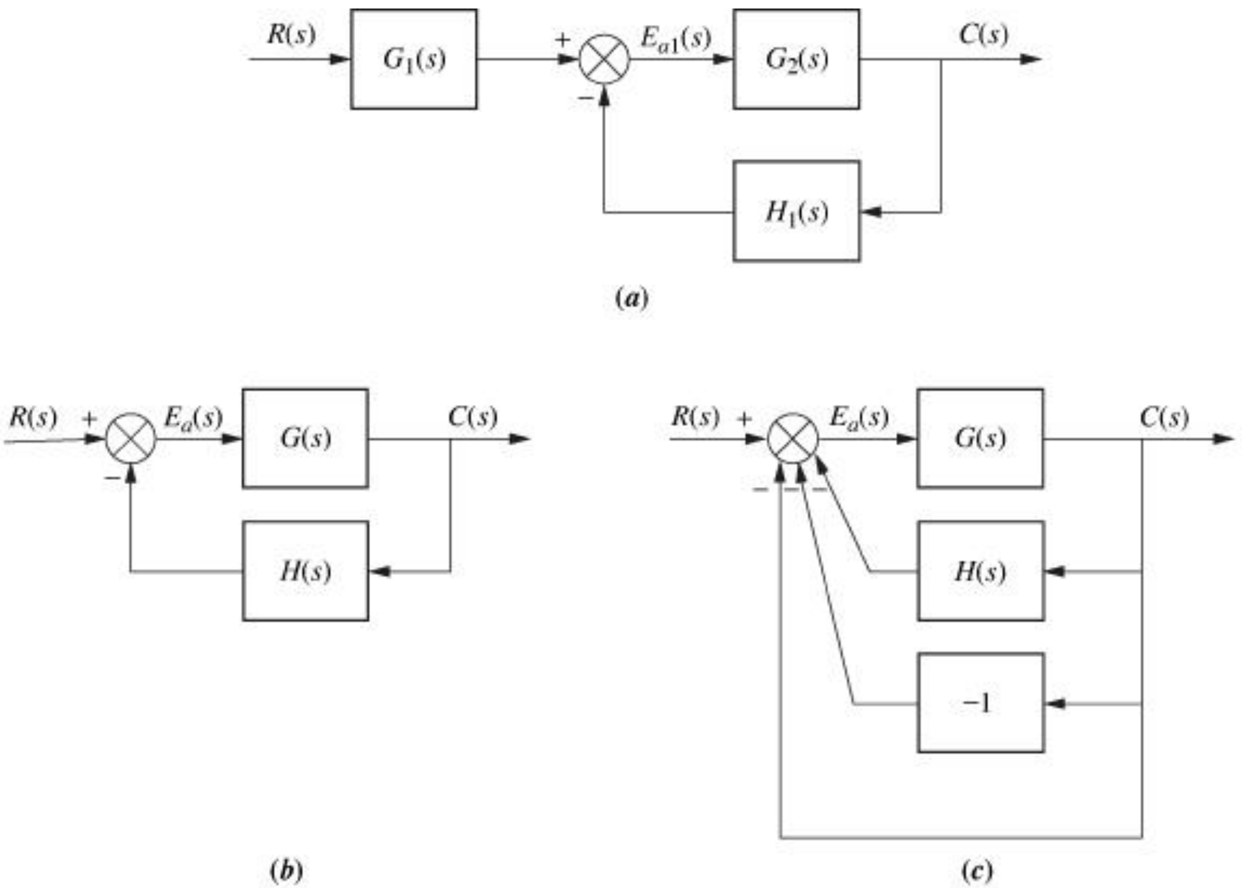
$$e_D(\infty) = -\lim_{s \to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

Steady-State Error for Disturbances

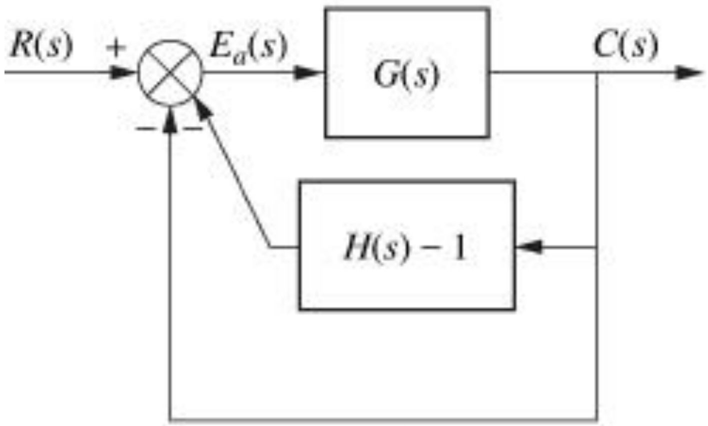


$$e_D(\infty) = -\frac{1}{\lim_{s\to 0}\frac{1}{G_2(s)} + \lim_{s\to 0} G_1(s)}$$

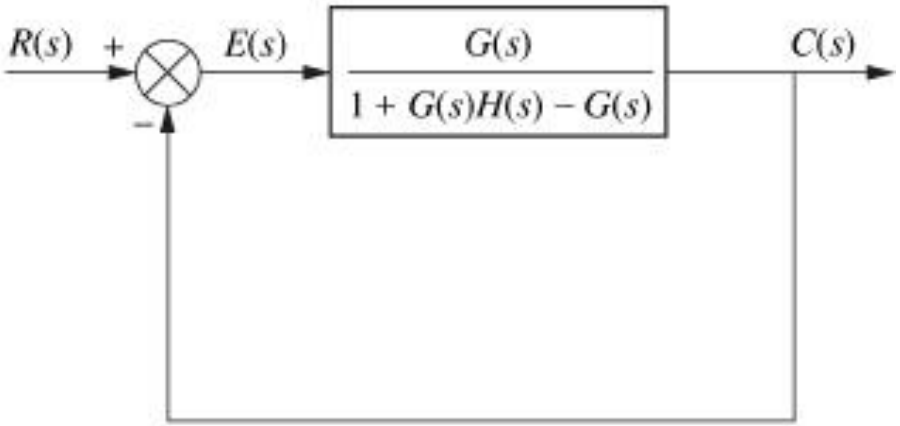
Steady-State Error for Nonunity Feedback Systems



Steady-State Error for Nonunity Feedback Systems

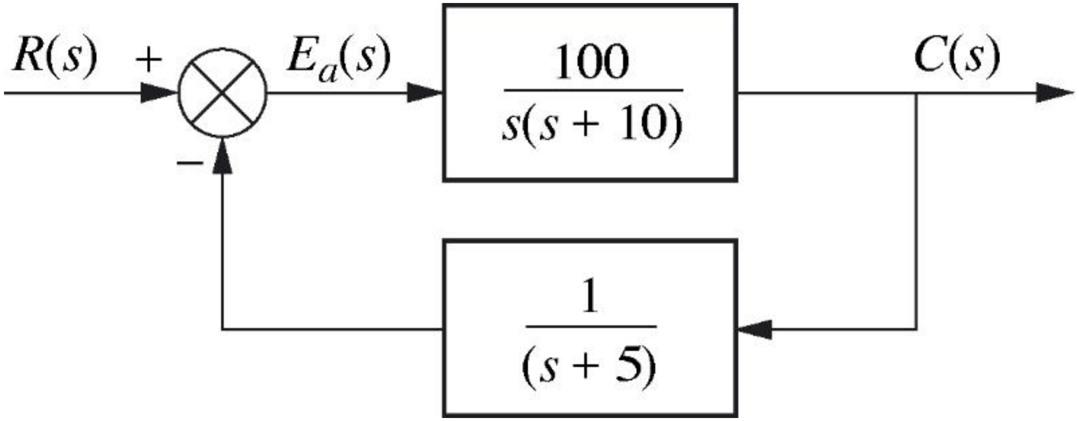


(d)



(e)

Steady-State Error for Nonunity Feedback Systems



$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)} = \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400}$$

Steady-State Error for Nonunity Feedback Systems

$$K_p = \lim_{s \to 0} G_e(s) = \frac{100 \times 5}{-400} = -\frac{5}{4}$$

The steady-state error, $e(\infty)$, is

$$e(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 - (5/4)} = -4$$

Sensiivity

$$S_{F:P} = \lim_{\Delta P \to 0} \frac{\text{Fractional change in the function, } F}{\text{Fractional change in the parameter, } P}$$

$$= \lim_{\Delta P \to 0} \frac{\Delta F / F}{\Delta P / P}$$

$$= \lim_{\Delta P \to 0} \frac{P}{F} \frac{\Delta F}{\Delta P}$$

which reduces to

$$S_{F:P} = \frac{P}{F} \frac{\delta F}{\delta P}$$