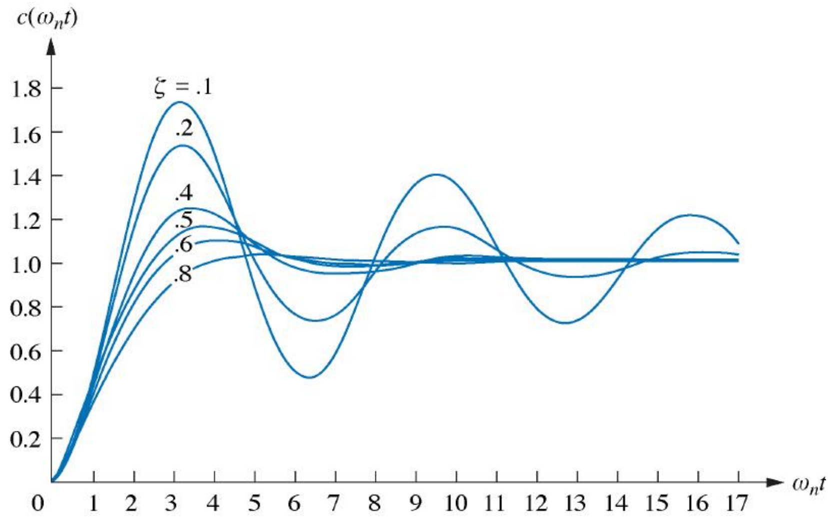
Time Response



System & Control Engineering Lab.
School of Mechanical Engineering

CHAPTER OBJECTIVES

- How to find the time response from the transfer function
- How to use poles and zeros to determine the response of a control system
- How to describe quantitatively the transient response of first- and second-order systems
- How to approximate higher-order systems as first or second order

POLES, ZEROS, AND SYSTEM RESPONSE

- POLES OF A TRANSFER FUNCTION
- ZEROS OF A TRANSFER FUNCTION
- FIRST-ORDER SYSTEMS
- SECOND-ORDER SYSTEMS: INTRODUCTION
- THE GENERAL SECOND-ORDER SYSTEM
- SYSTEM RESPONSE WITH ADDITIONAL POLES
- SYSTEM RESPONSE WITH ZEROS

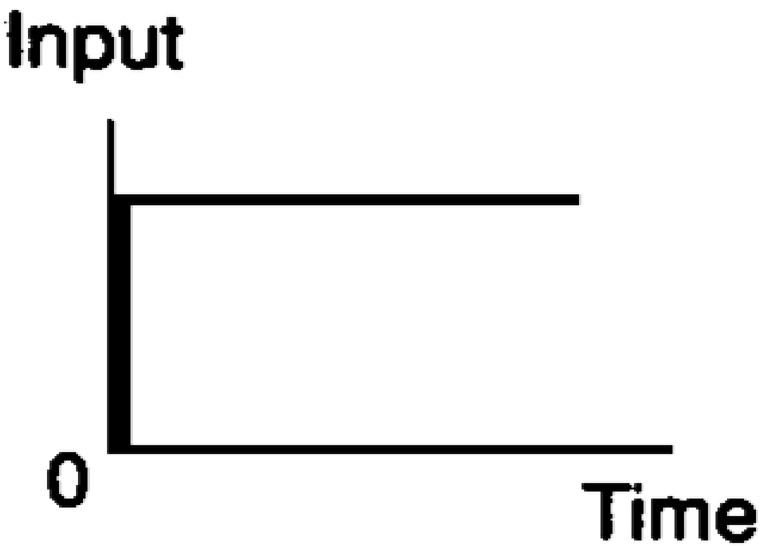
The Performance of Control Systems

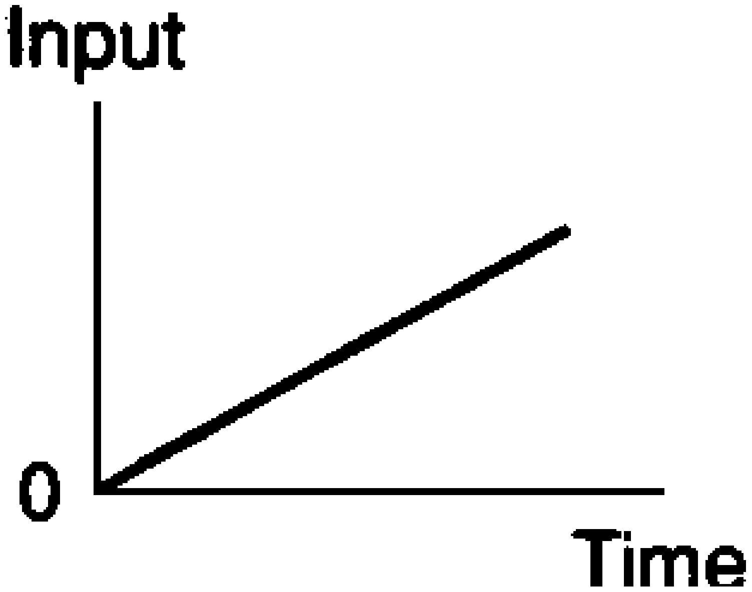
- Test input signals
- Transient response and Steady-state response
- Absolute stability, relative stability
- Steady-state error

Typical Test Signal

- Step functions
- Ramp functions
- Acceleration functions
- Impulse functions
- Sinusoidal functions
- White noise signals

Typical Test Signal





Step functions

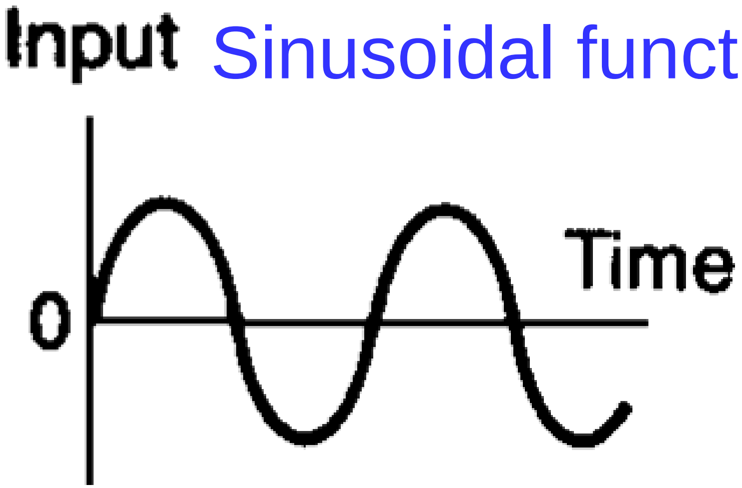
Ramp functions

$$f(t) = 1(t);\ F(s) = \frac{1}{s}$$

$$f(t) = t;\ F(s) = \frac{1}{s^2}$$

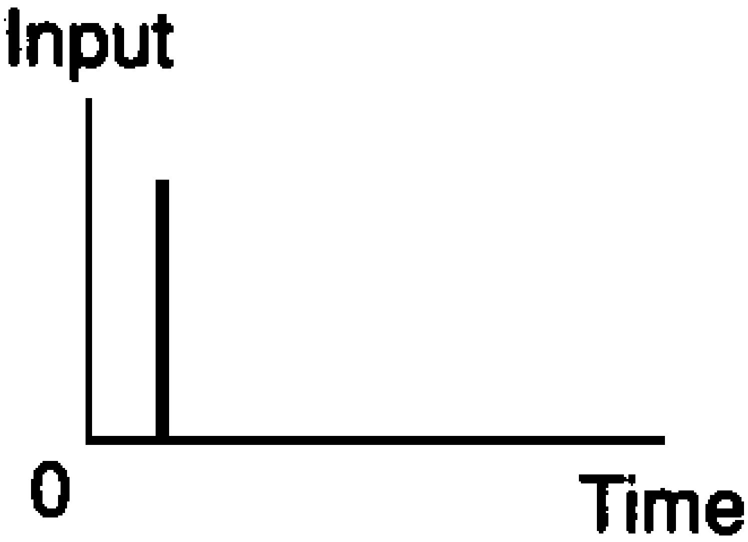
Typical Test Signal

Sinusoidal functions



$$f(t) = \sin \omega t;\ F(s) = \frac{\omega}{s^2 + \omega^2}$$

$$f(t) = \cos \omega t;\ F(s) = \frac{s}{s^2 + \omega^2}$$



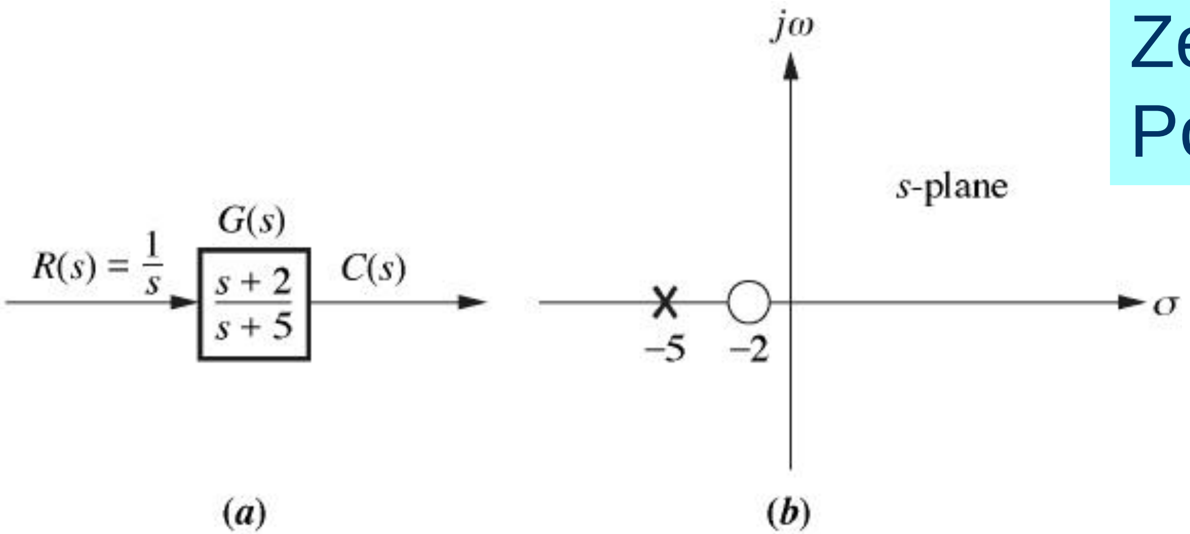
Impulse functions

$$\text{Unit impulse} = \delta(\text{t});\ F(s) = 1$$

POLES AND ZEROS OF A FIRST-ORDER SYSTEM: AN EXAMPLE

$$\frac{C(s)}{R(s)} = G(s) = \frac{s+2}{s+5} \Rightarrow C(s) = \frac{s+2}{s+5}R(s) \Rightarrow C(s) = \left(\frac{s+2}{s+5}\right)\frac{1}{s}$$

$R(s)$ is the unit-step input



Zero = -2
Pole = -5

POLES AND ZEROS OF A FIRST-ORDER SYSTEM: AN EXAMPLE

$R(s)$ is the unit-step input

$$C(s)=\frac{s+2}{s+5}R(s) \Longrightarrow C(s)=\left(\frac{s+2}{s+5}\right)\frac{1}{s}$$

$$C(s)=\frac{(s+2)}{s(s+5)}=\frac{A}{s}+\frac{B}{s+5}=\frac{2/5}{s}+\frac{3/5}{s+5} \quad (4.1)$$

where

$$A=\left.\frac{(s+2)}{(s+5)}\right|_{s\to 0}=\frac{2}{5}$$

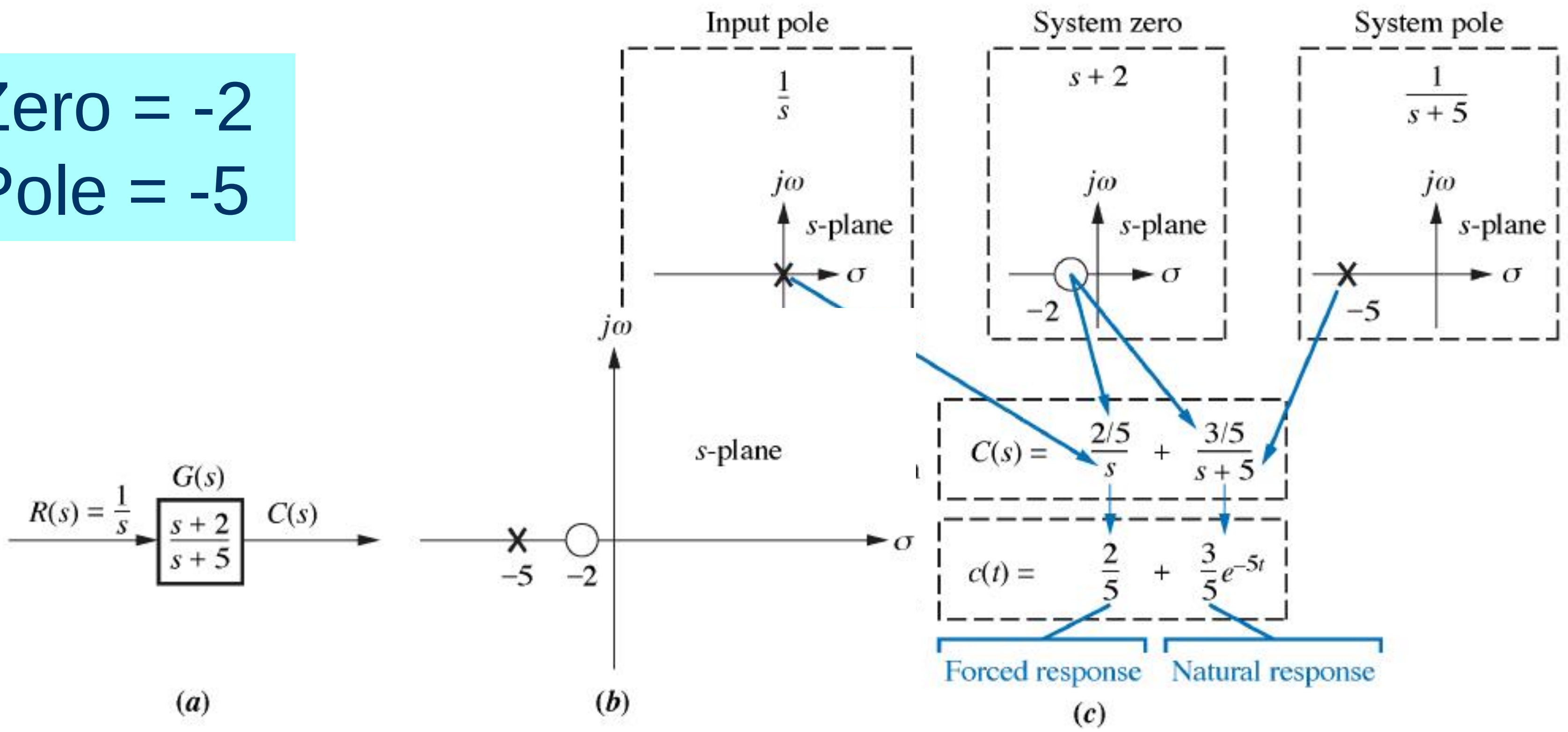
$$B=\left.\frac{(s+2)}{s}\right|_{s\to -5}=\frac{3}{5}$$

Thus,

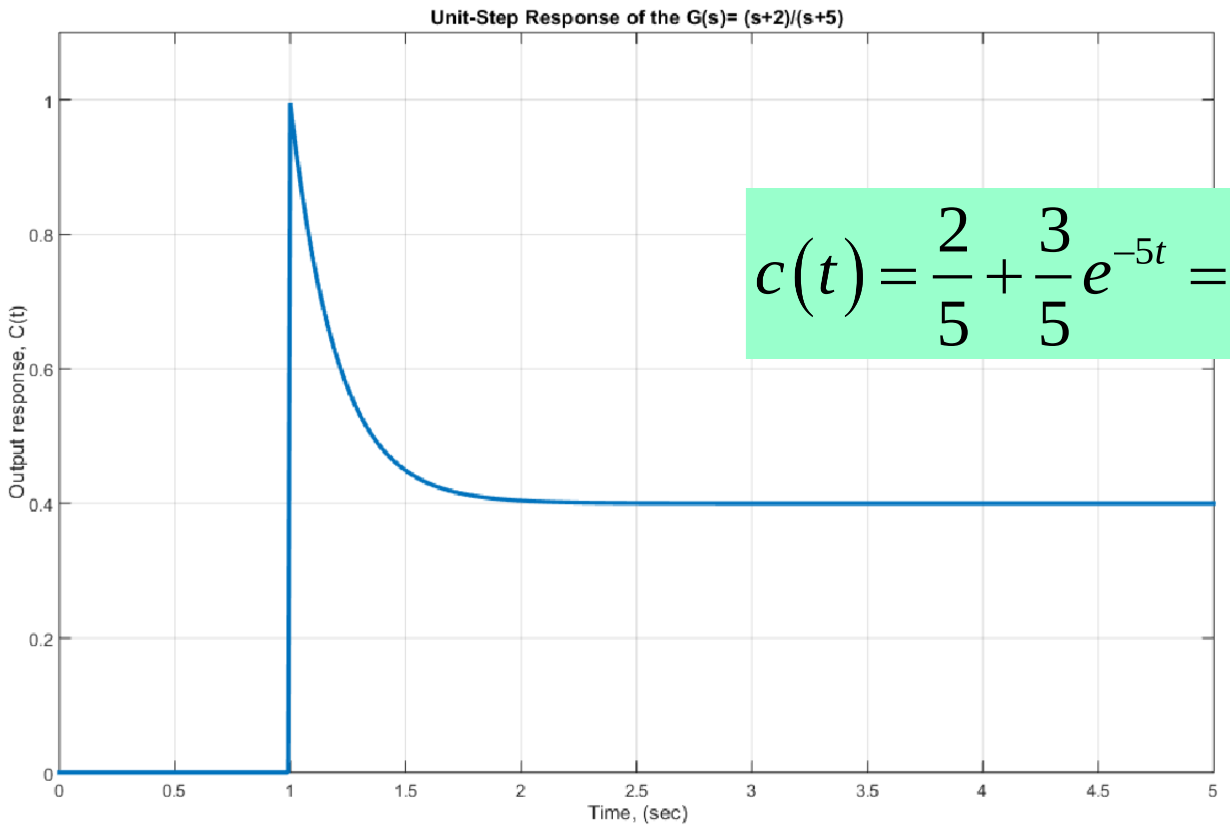
$$c(t)=\frac{2}{5}+\frac{3}{5}e^{-5t} \quad (4.2)$$

POLES AND ZEROS OF A FIRST-ORDER SYSTEM: AN EXAMPLE

Zero = -2
Pole = -5

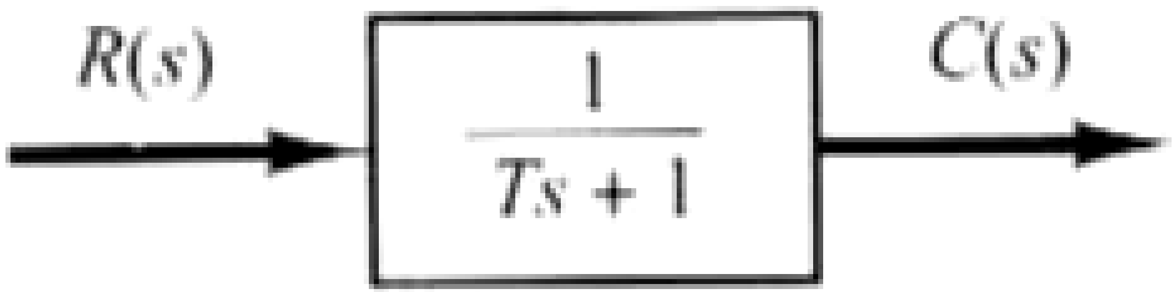


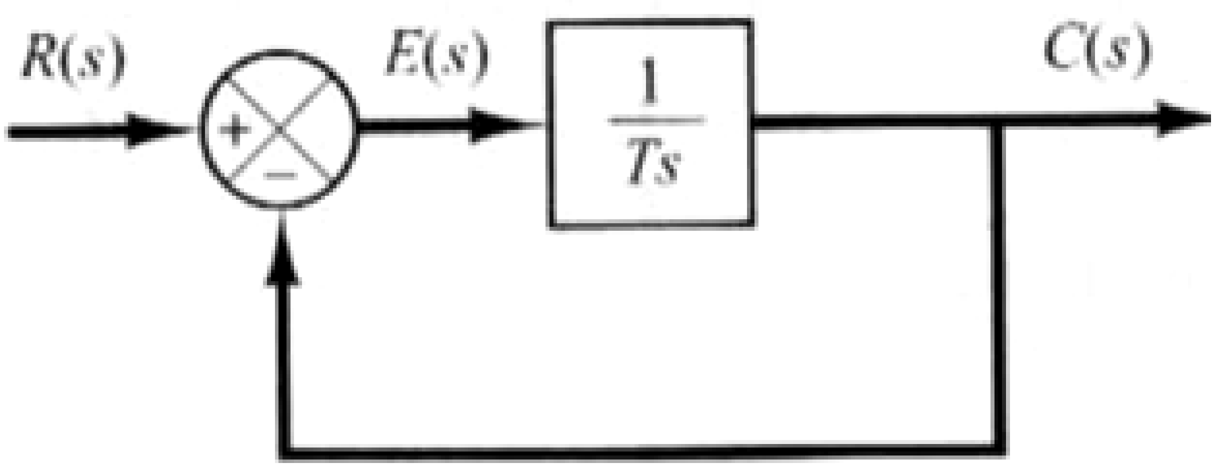
POLES AND ZEROS OF A FIRST-ORDER SYSTEM: AN EXAMPLE



$$c(t)=\frac{2}{5}+\frac{3}{5}e^{-5t}=0.4+0.6e^{-5t}$$

FIRST-ORDER SYSTEMS





Unit-Step Response of First-Order Systems

Transfer function of the first-order system

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1} \rightarrow C(s) = \frac{1}{Ts+1}R(s)$$

For unit-step function $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s}$$

Unit-Step Response of First-Order Systems

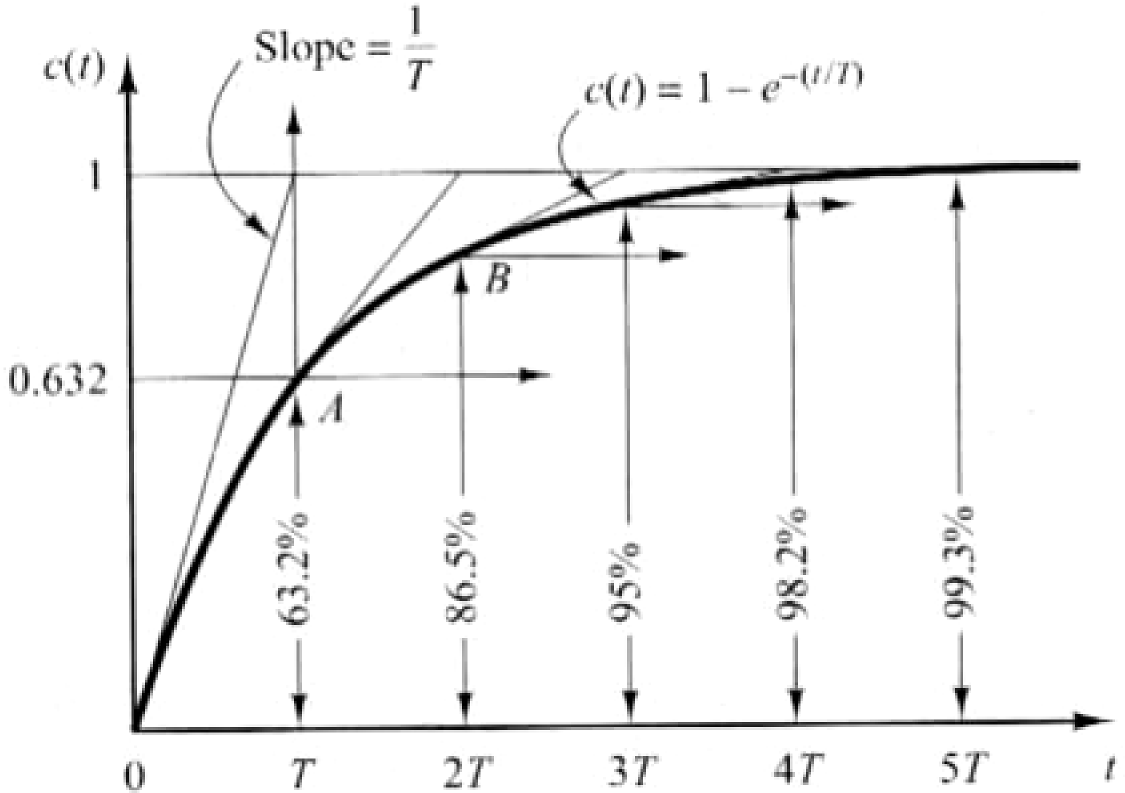
Inverse Laplace transform

$$c(t) = 1 - e^{-t/T} \quad \text{for } t \geq 0$$

C(t) = Force response + Free response

and t=T $$c(t) = 1 - e^{-1} = 0.632$$

Unit-Step Response of First-Order Systems



Unit-Step response

FIRST-ORDER SYSTEMS

- TIME CONSTANT (T)
- RISE TIME, Tr
- SETTLING TIME, Ts

$$T_r = 2.2T$$

$$T_s = 4T \quad , 2\% \text{ error}$$

$$T_s = 3T \quad , 5\% \text{ error}$$

Second-Order Systems

In transfer function (Laplace Transform)

$$C(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s)$$

$$\frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = G(s)$$

Second-Order Systems

1. Overdamped $\zeta > 1$
2. Critically damped $\zeta = 1$
3. Undamped $\zeta = 0$
4. Underdamped $0 < \zeta < 1$

Second-Order Systems: Unit-Step Input

1. *Overdamped responses*

   $$\sigma_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

   Poles: Two real at $-\sigma_1, -\sigma_2$

   Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

   $$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

2. *Underdamped responses*

   $$\sigma_d = -\zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

   Poles: Two complex at $-\sigma_d \pm j\omega_d$

   Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

   $$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$

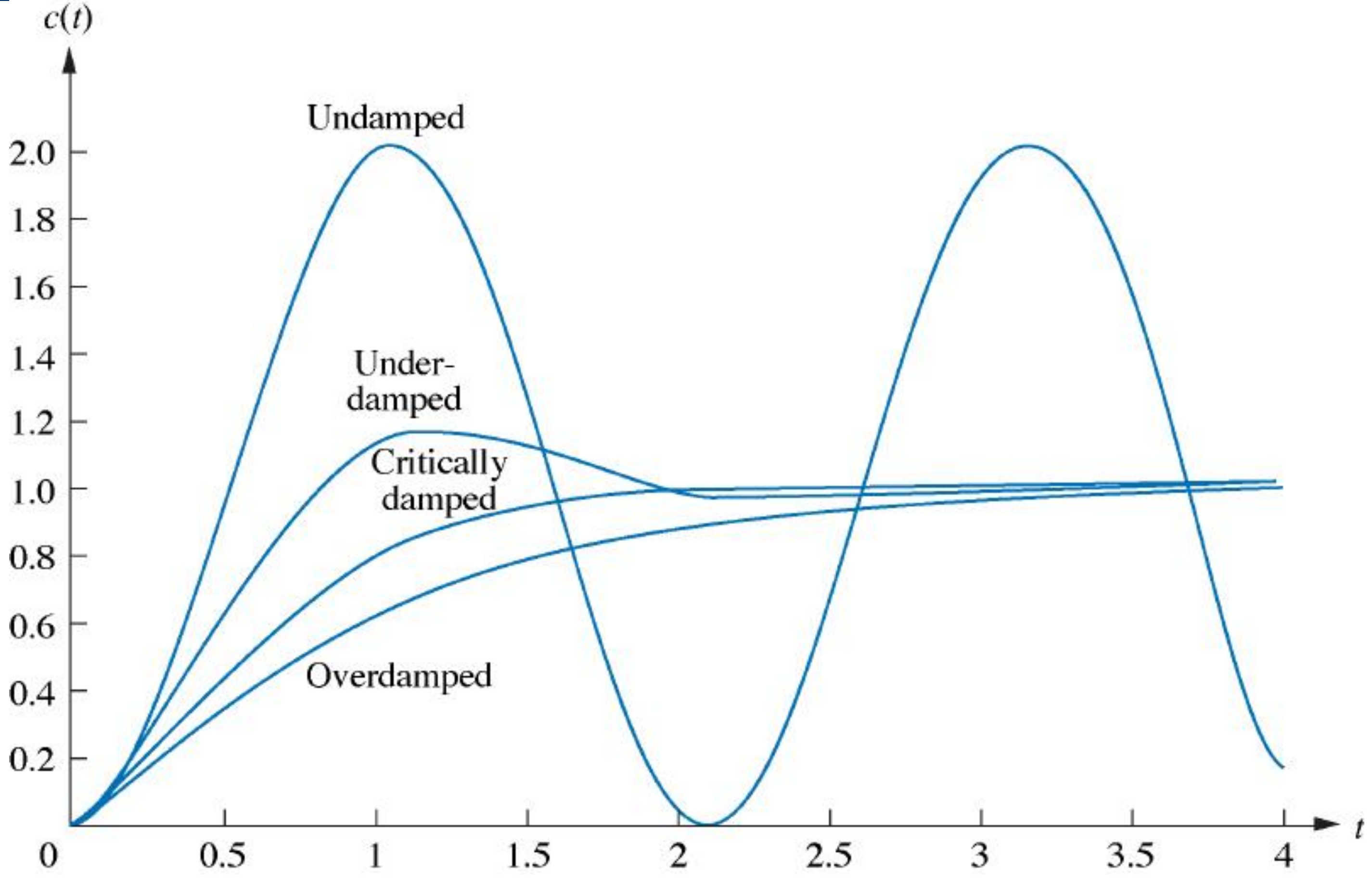
3. *Undamped responses*

   Poles: Two imaginary at $\pm j\omega_1$

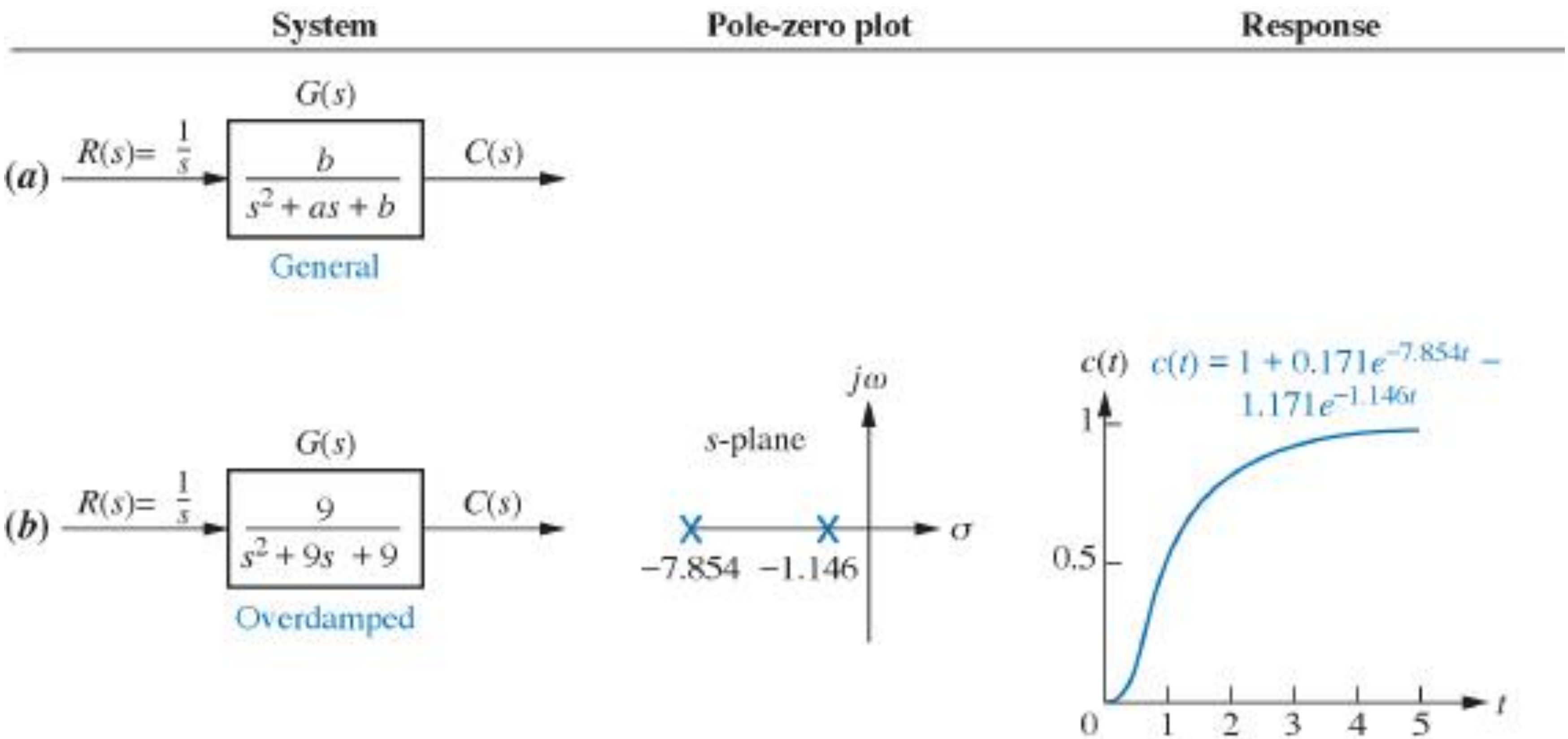
   Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

   $$c(t) = A\cos(\omega_1 t - \phi)$$

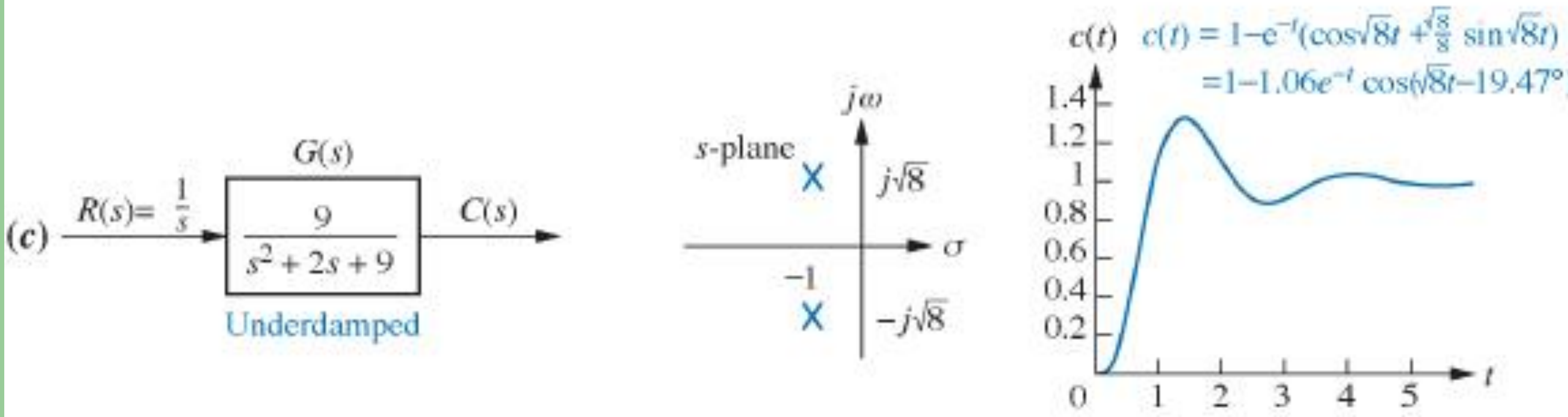
Second-Order Systems: Unit-Step Input



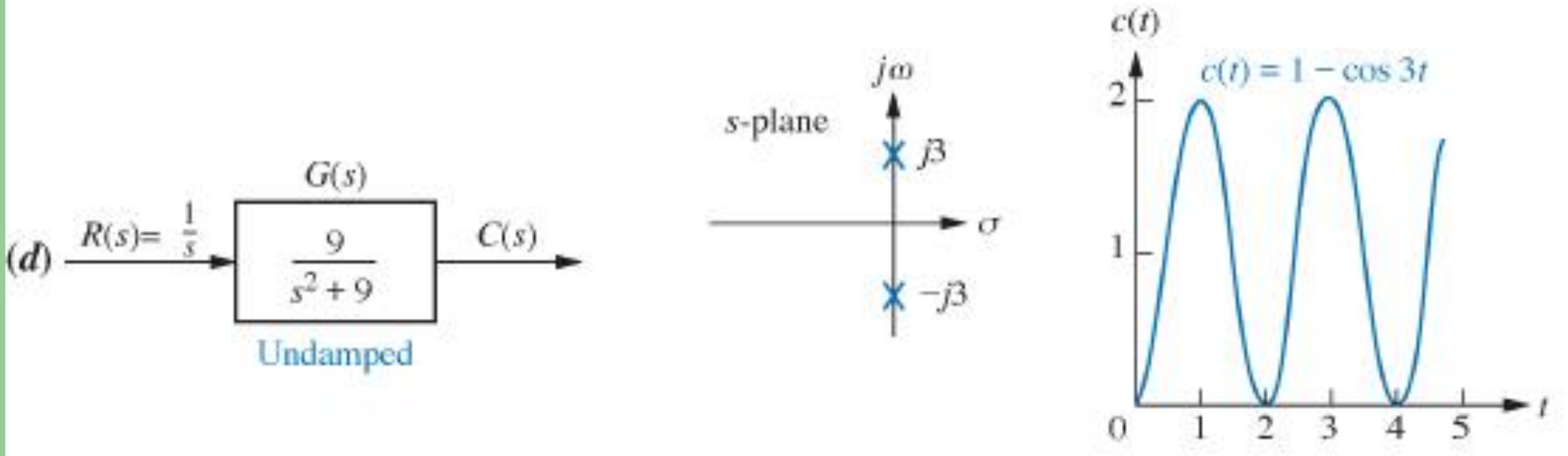
Second-Order Systems: Unit-Step Input



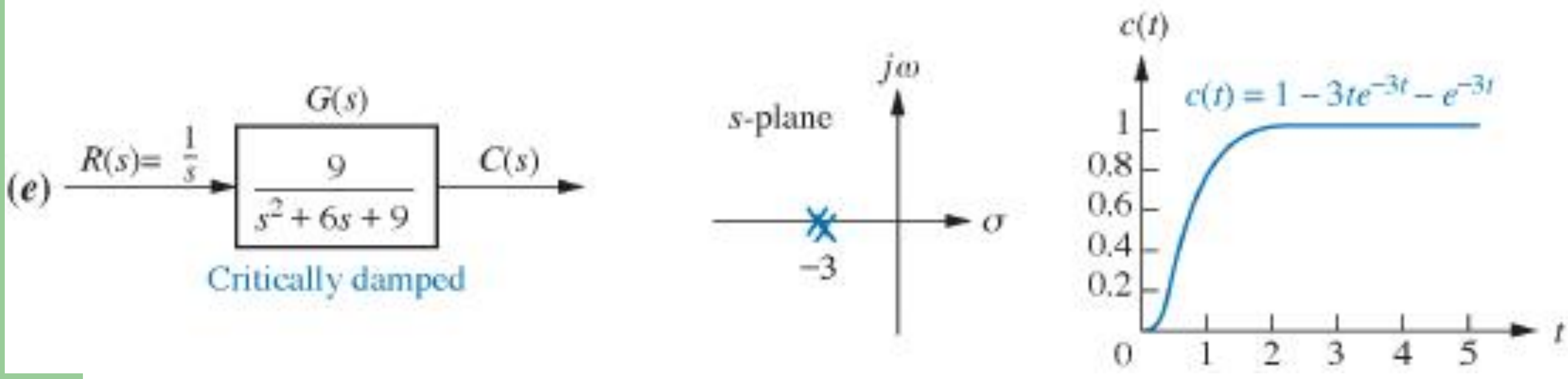
Second-Order Systems: Unit-Step Input



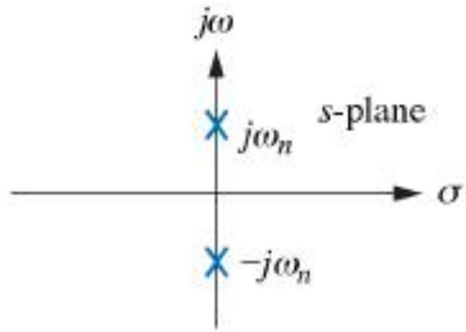
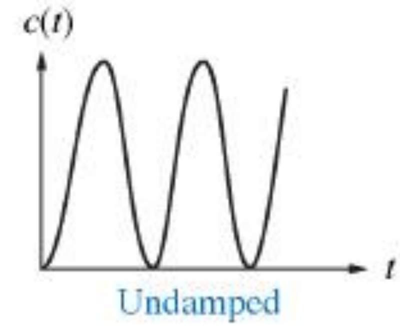
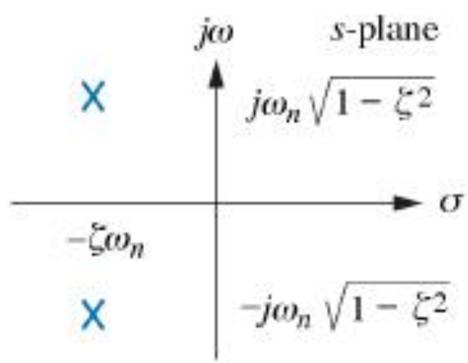
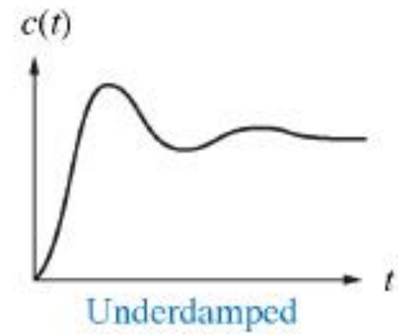
Second-Order Systems: Unit-Step Input



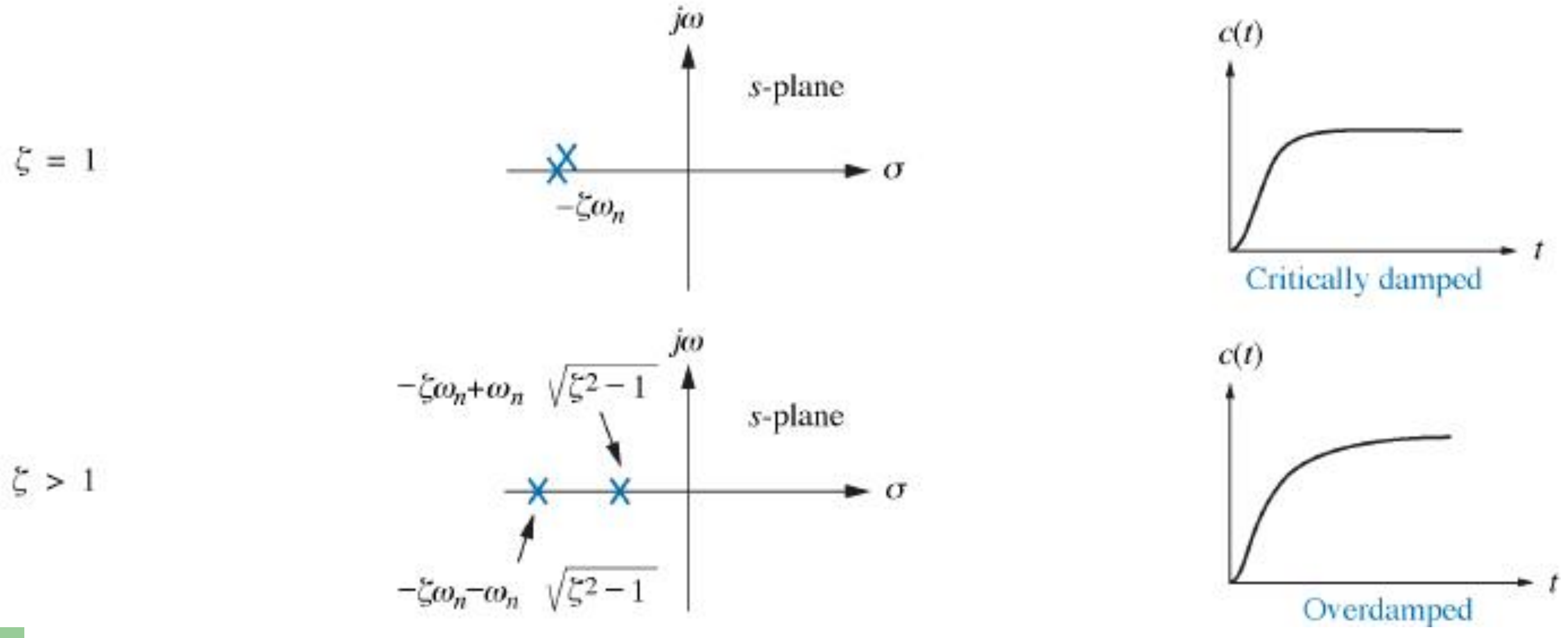
Second-Order Systems: Unit-Step Input



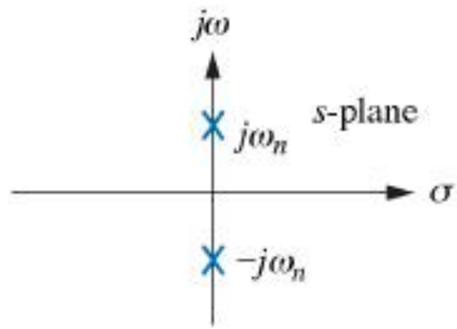
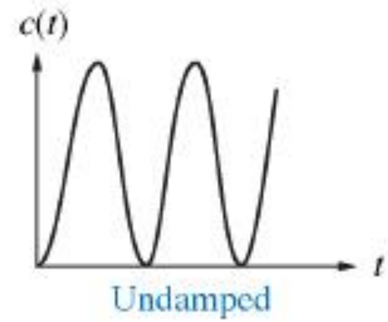
Second-Order Systems: Unit-Step Input

ζ	Poles	Step response
0		
$0 < \zeta < 1$		

Second-Order Systems: Unit-Step Input



Second-Order Systems: Unit-Step Input

ζ	Poles	Step response
0		

Underdamped Second-Order Systems Response

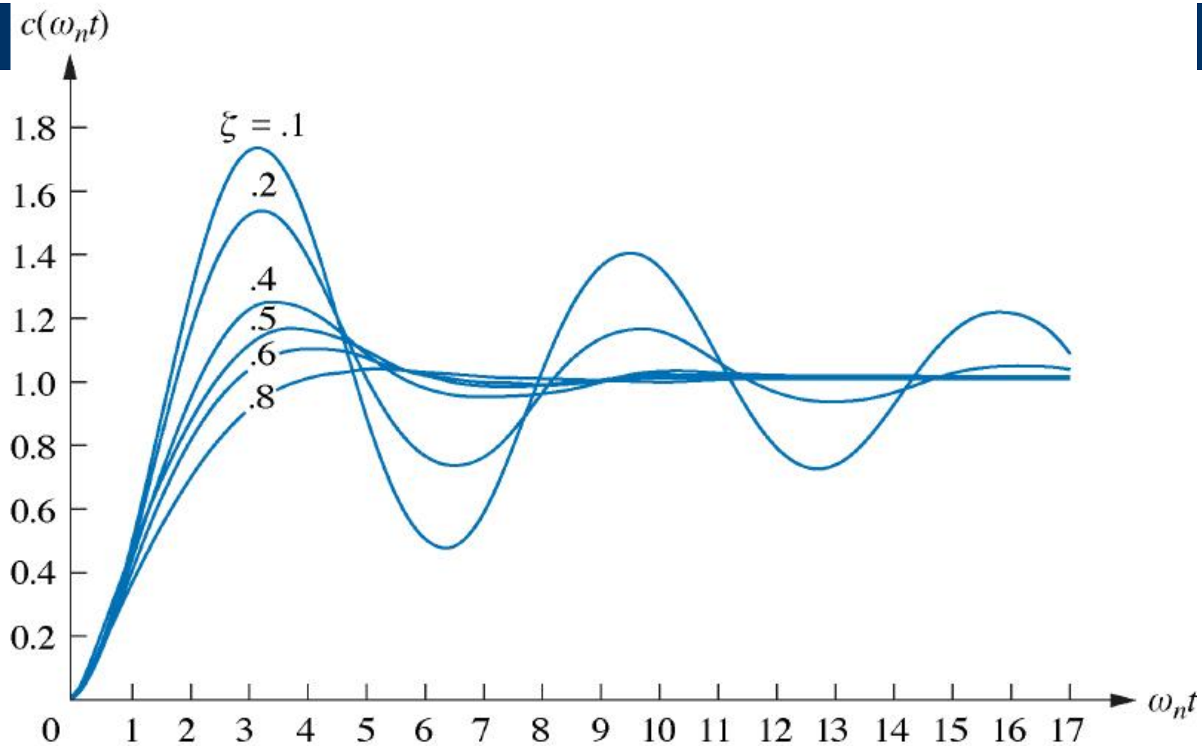
Let us begin by finding the step response for the general second-order system of Eq. (4.22). The transform of the response, $C(s)$, is the transform of the input times the transfer function, or

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.26}$$

where it is assumed that $\zeta < 1$ (the underdamped case). Expanding by partial fractions, using the methods described in Section 2.2, Case 3, yields

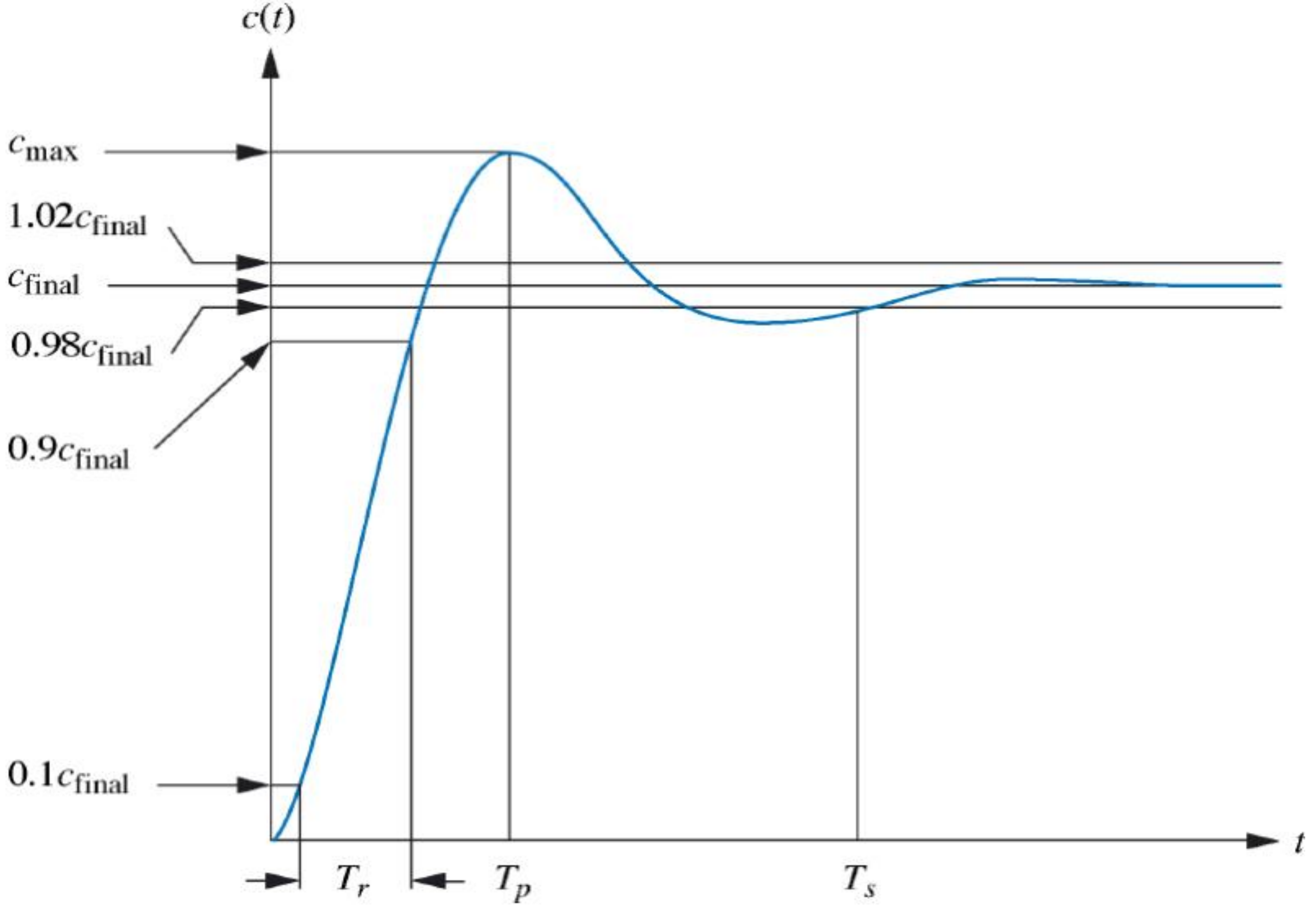
$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \tag{4.27}$$

Underdamped Second-Order Systems Response



$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right)$$
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t - \phi\right) \quad (4.28)$$

Underdamped Second-Order Systems Response



Underdamped Second-Order Systems Response

1. *Rise time*, T_r. The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
2. *Peak time*, T_P. The time required to reach the first, or maximum, peak.
3. *Percent overshoot*, *%OS*. The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.
4. *Settling time*, T_s. The time required for the transient's damped oscillations to reach and stay within ±2% of the steady-state value.

Underdamped Second-Order Systems Response

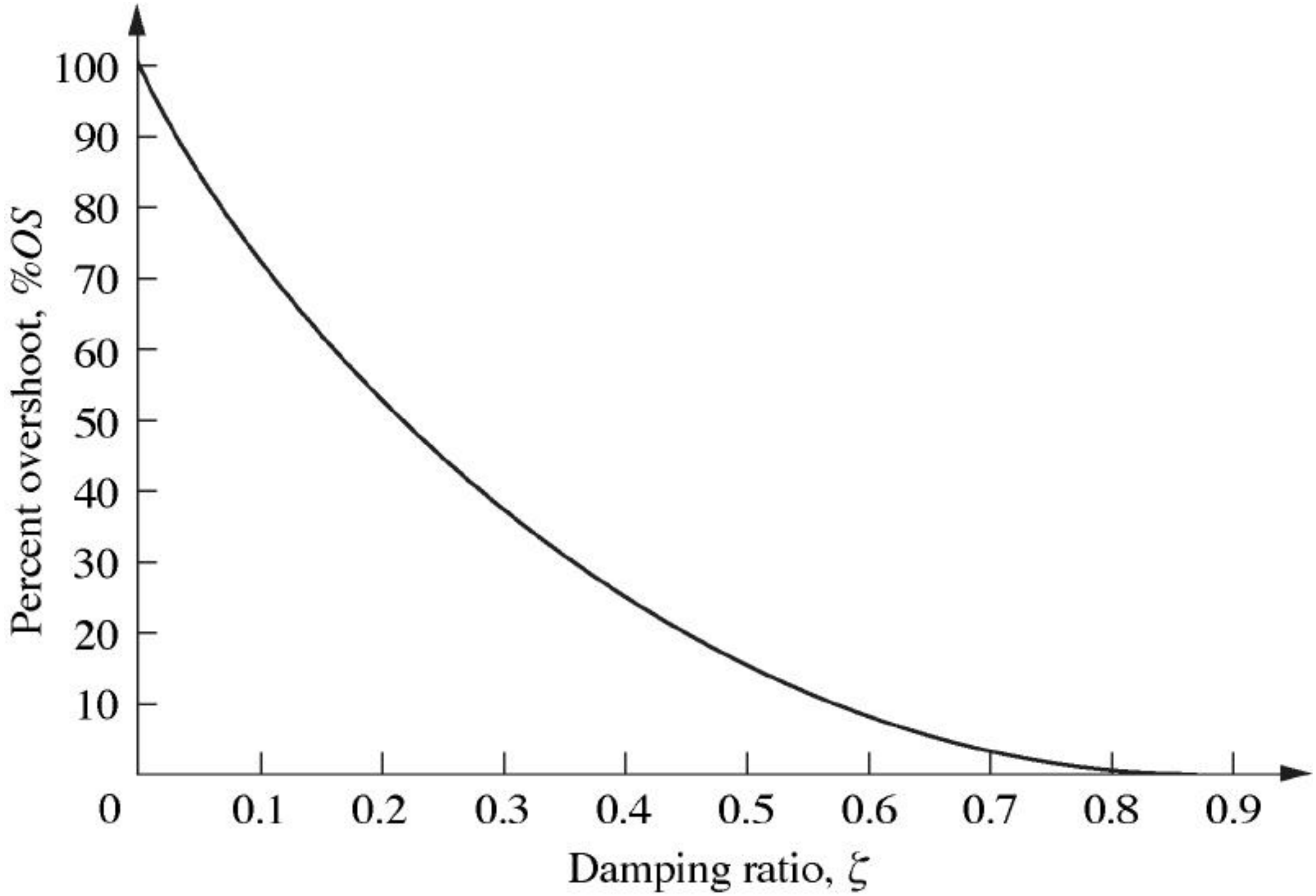
$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

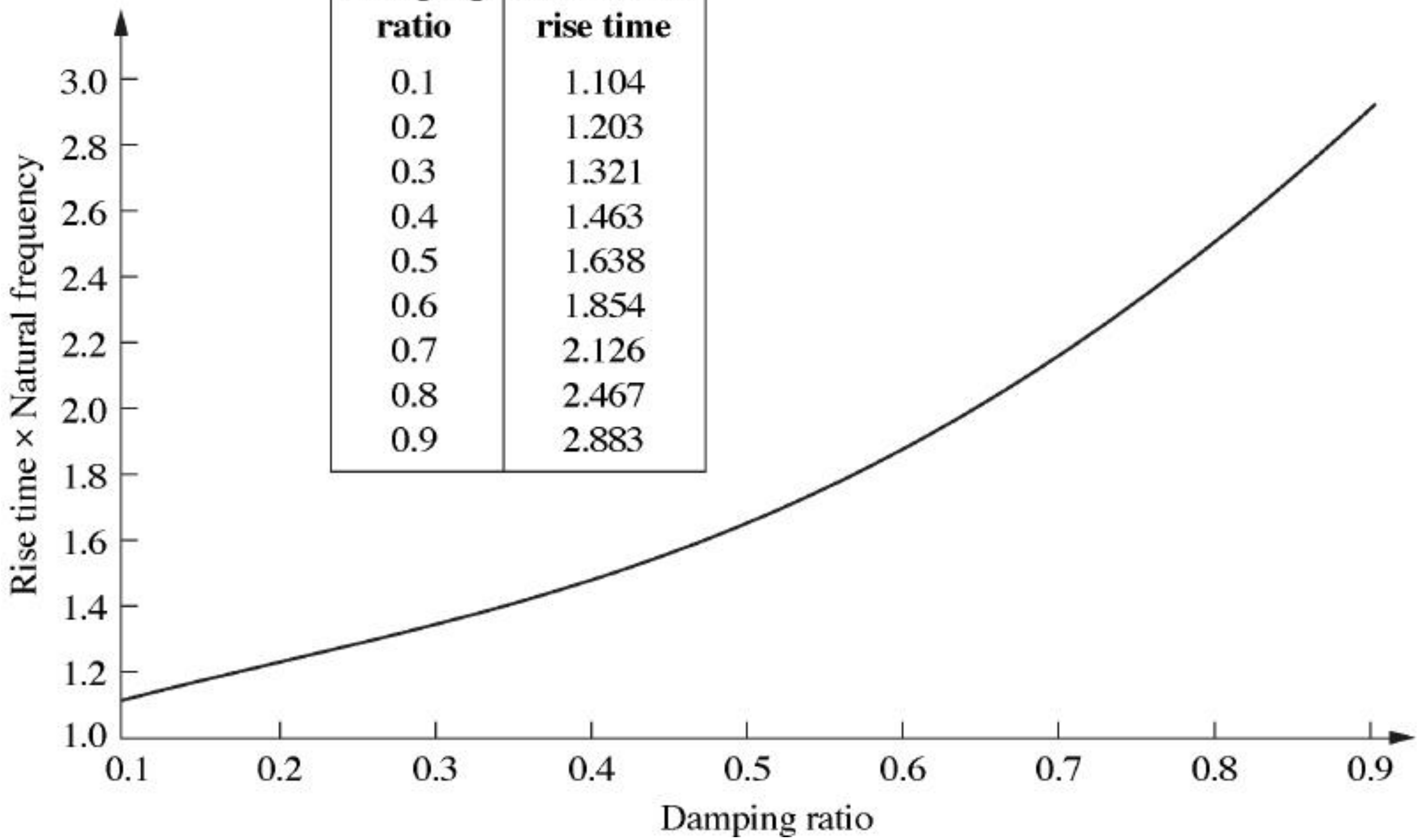
$$T_s = \frac{4}{\zeta\omega_n}$$

Underdamped Second-Order Systems Response

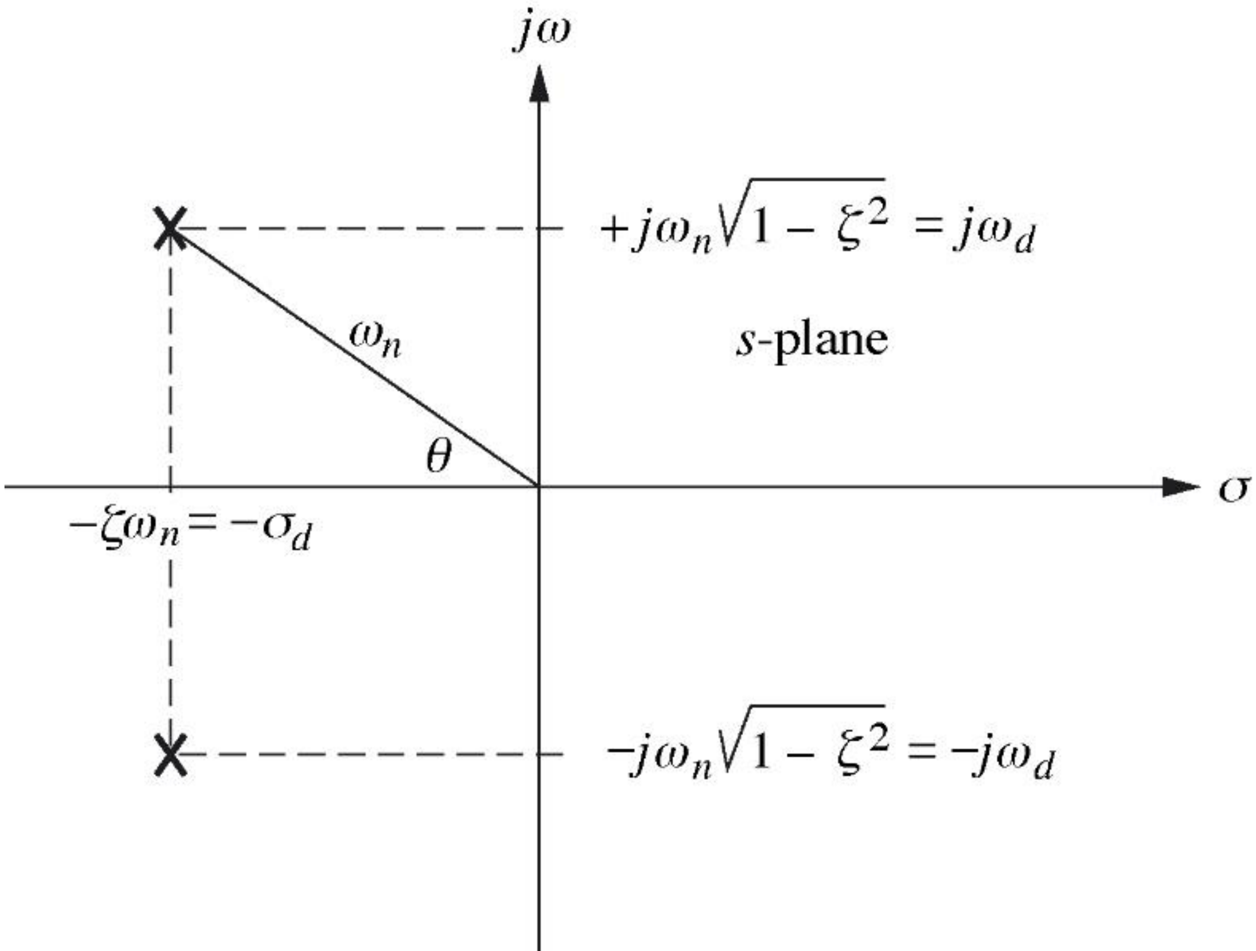


Underdamped Second-Order Systems Response

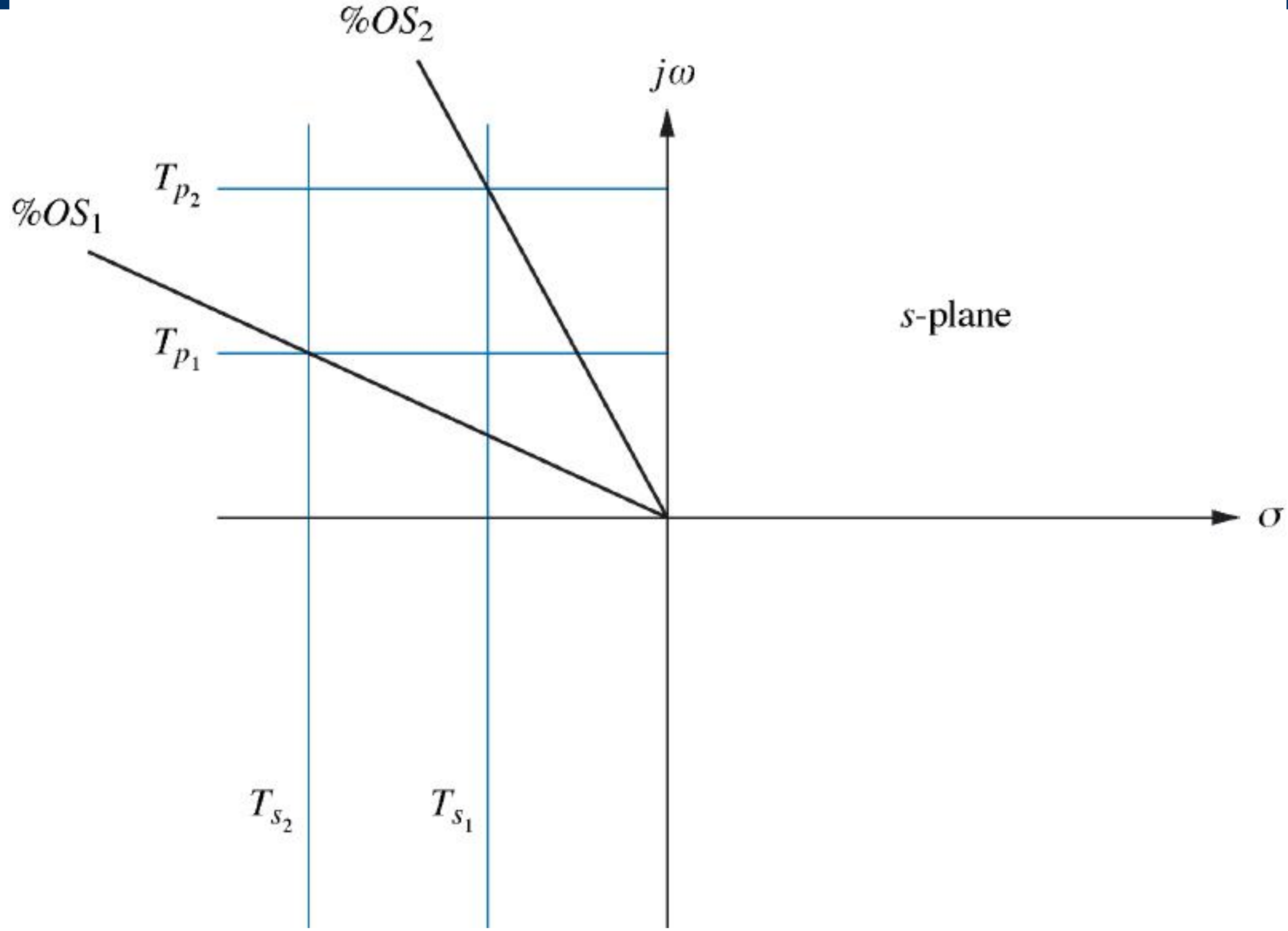
Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883



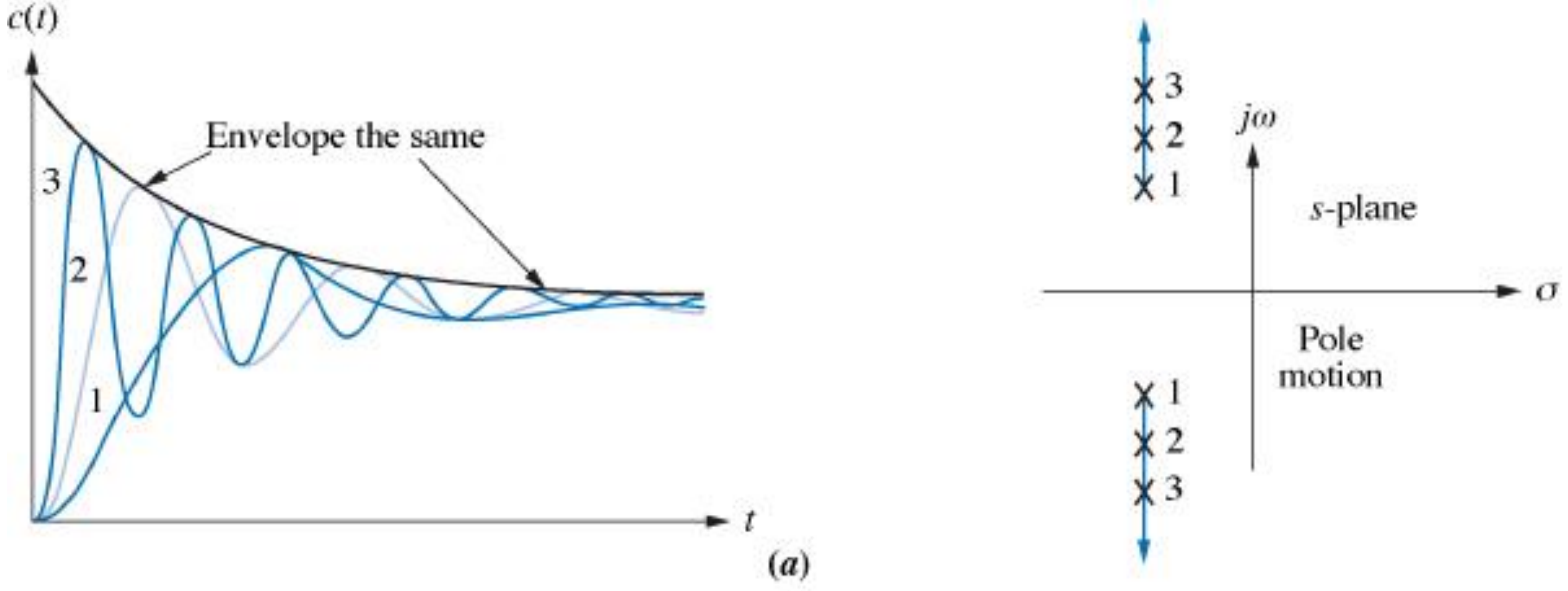
Underdamped Second-Order Systems Response



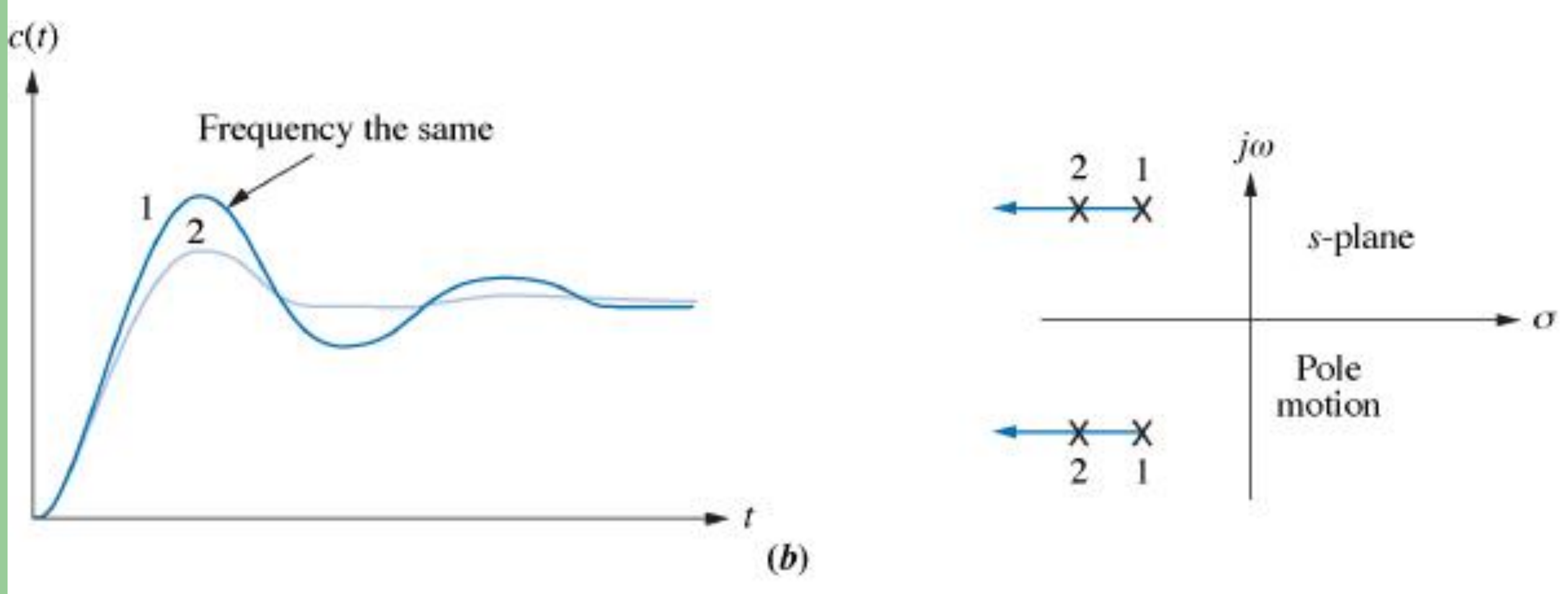
Underdamped Second-Order Systems Response



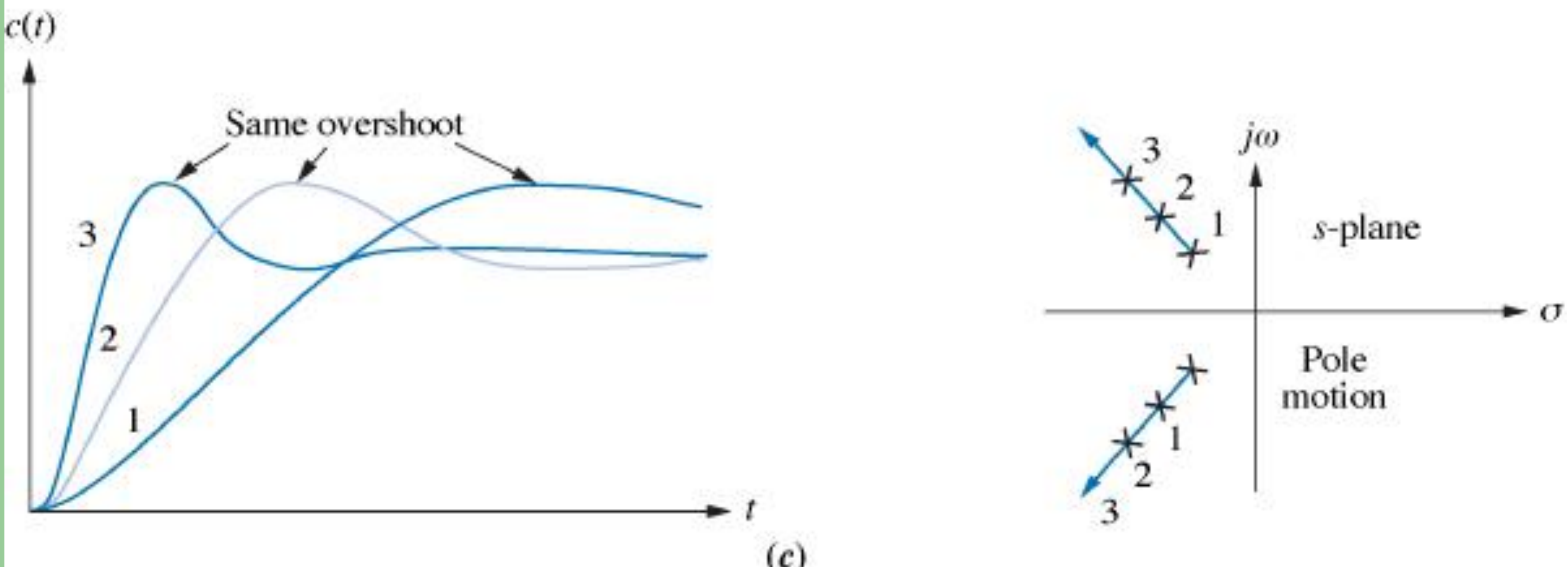
Underdamped Second-Order Systems Response



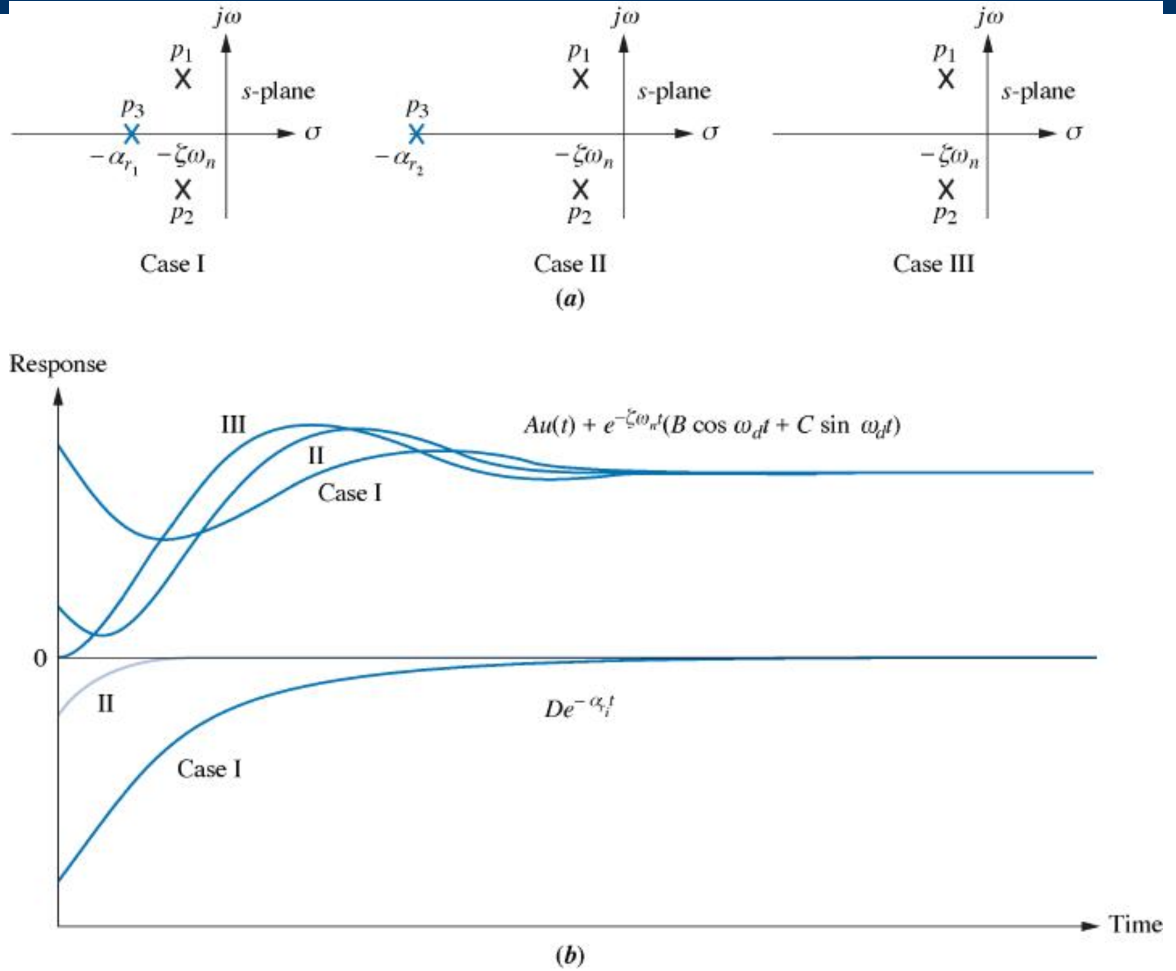
Underdamped Second-Order Systems Response



Underdamped Second-Order Systems Response



SYSTEM RESPONSE WITH ADDITIONAL POLES



SYSTEM RESPONSE WITH ADDITIONAL ZEROS