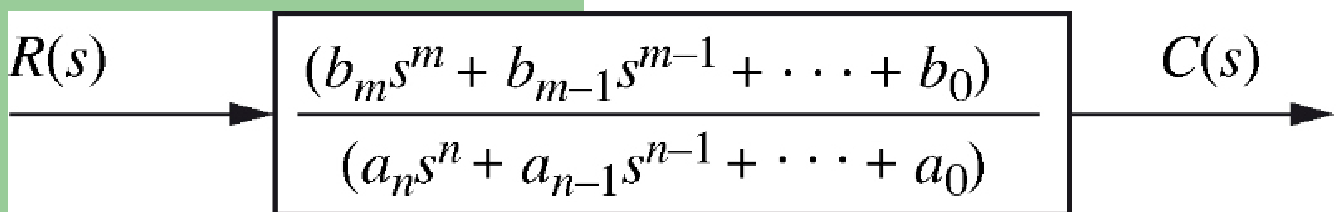
Modeling in the Frequency Domain

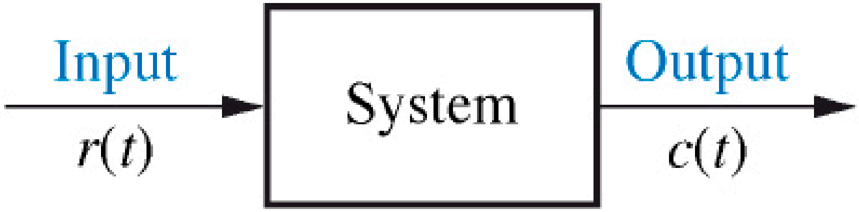


System & Control Engineering Lab.
School of Mechanical Engineering

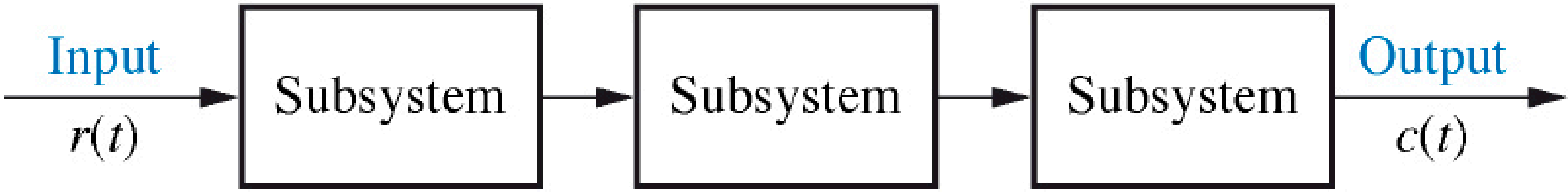
CHAPTER OBJECTIVES

- Review the Laplace transform
- Learn how to find a mathematical model, called a transfer function, for linear, time-invariant electrical, mechanical, and electromechanical systems
- Learn how to linearize a nonlinear system in order to find the transfer function

Block Diagram



(*a*)



(*b*)

Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

LAPLACE TRANSFORM REVIEW

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt \qquad (2.1)$$

TABLE 2.1 Laplace transform table

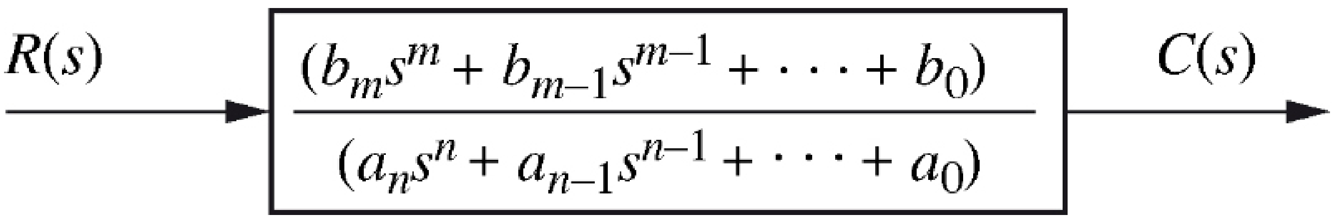
Item no.	$f(t)$	$F(s)$
1.	$\delta(l)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$l^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

LAPLACE TRANSFORM REVIEW

TABLE 2.2 Laplace transform theorems

Item no.	Theorem		Name
1.	$\mathscr{L}[f(t)] = F(s)$	$= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)]$	$= kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)]$	$= F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)]$	$= F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)]$	$= e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)]$	$= \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right]$	$= sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - f(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{1} f(\tau)d\tau\right]$	$= \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty)$	$= \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+)$	$= \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

TRANSFER FUNCTIONS



$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}$$

TRANSFER FUNCTIONS OF PHYSICAL SYSTEMS

- ELECTRICAL NETWORK TRANSFER FUNCTIONS
- TRANSLATIONAL MECHANICAL SYSTEM TRANSFER FUNCTIONS
- ROTATIONAL MECHANICAL SYSTEM TRANSFER FUNCTIONS
- TRANSFER FUNCTIONS FOR SYSTEMS WITH GEARS
- ELECTROMECHANICAL SYSTEM TRANSFER FUNCTIONS
- ELECTRIC CIRCUIT ANALOGS

ELECTRICAL NETWORK TRANSFER FUNCTIONS

- **Part-I:** Electrical Systems
  - Basic Elements of Electrical Systems
  - Equations for Basic Elements
  - Examples
- **Part-II:** Electronic Systems
  - Operational Amplifiers
  - Inverting *vs* Non-inverting
  - Examples

Basic Elements of Electrical Systems

Symbol →

- The time domain expression relating voltage and current for the resistor is given by Ohm's law i-e

$$v_R(t) = i_R(t)R$$

- The Laplace transform of the above equation is

$$V_R(s) = I_R(s)R$$

Basic Elements of Electrical Systems

- The time domain expression relating voltage and current for the Capacitor is given as:

$$v_c(t) = \frac{1}{C}\int i_c(t)dt$$

- The Laplace transform of the above equation (assuming there is no charge stored in the capacitor) is

$$V_c(s) = \frac{1}{Cs} I_c(s)$$

Basic Elements of Electrical Systems

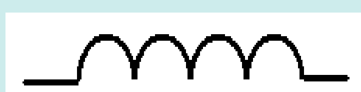
- The time domain expression relating voltage and current for the inductor is given as:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

- The Laplace transform of the above equation (assuming there is no energy stored in inductor) is

$$V_L(s) = LsI_L(s)$$

V-I and I-V relations

Component	Symbol	V-I Relation	I-V Relation
Resistor		$v_R(t) = i_R(t)R$	$i_R(t) = \frac{v_R(t)}{R}$
Capacitor		$v_c(t) = \frac{1}{C}\int i_c(t)dt$	$i_c(t) = C\frac{dv_c(t)}{dt}$
Inductor		$v_L(t) = L\frac{di_L(t)}{dt}$	$i_L(t) = \frac{1}{L}\int v_L(t)dt$

ELECTRICAL NETWORK TRANSFER FUNCTIONS

TABLE 2.3 **Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors**

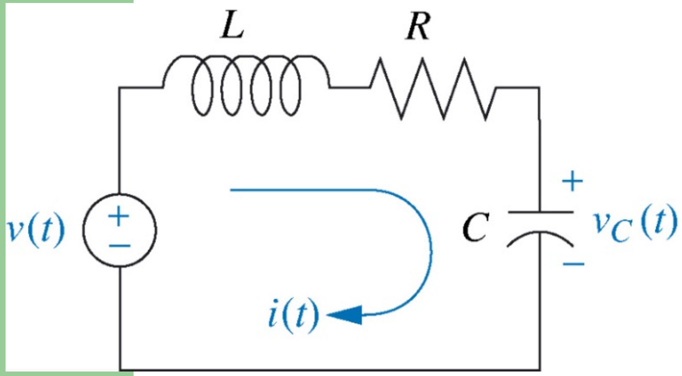
Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)\, d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t) = \mathrm{V}$ (volts), $i(t) = \mathrm{A}$ (amps), $q(t) = \mathrm{Q}$ (coulombs), $C = \mathrm{F}$ (farads), $R = \Omega$ (ohms), $G = \Omega$ (mhos), $L = H$ (henries).

[2] *Passive* means that there is no internal source of energy.

ELECTRICAL NETWORK TRANSFER FUNCTIONS

Kirchhoff's Voltage Law (KVL), Loop method



$$\sum v_{drop} = 0$$

$$v_L + v_R + v_C - v = 0$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t)$$

$$i(t) = dq(t)/dt$$

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t)$$

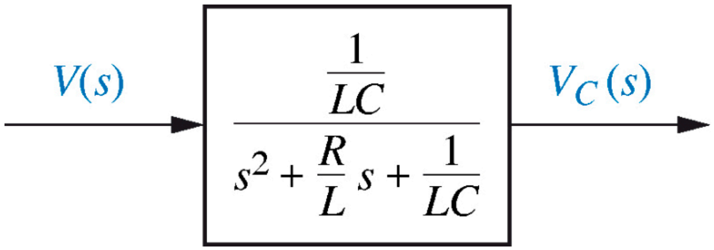
ELECTRICAL NETWORK TRANSFER FUNCTIONS

$$q(t) = Cv_C(t)$$

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t)$$

$$(LCs^2 + RCs + 1)V_C(s) = V(s)$$

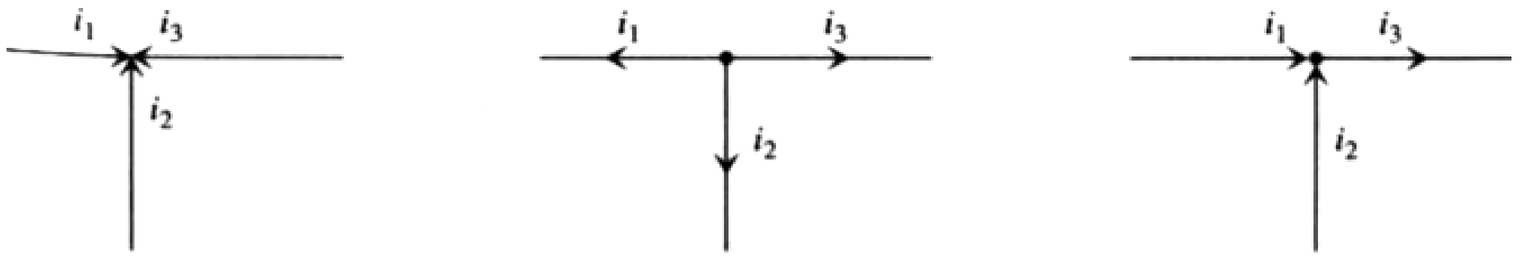
$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



ELECTRICAL NETWORK TRANSFER FUNCTIONS

Kirchhoff’s Current Law (KCL), Node method

$$\sum_k (i_k)_{in} = 0$$



(a) (b) (c)

$$i_1 + i_2 + i_3 = 0 \qquad -(i_1 + i_2 + i_3) = 0 \qquad i_1 + i_2 - i_3 = 0$$

ELECTRICAL NETWORK TRANSFER FUNCTIONS

Impedance Method

$$\frac{V(s)}{I(s)} = Z(s)$$

For the capacitor,

$$V(s) = \frac{1}{Cs}I(s)$$

For the resistor,

$$V(s) = RI(s)$$

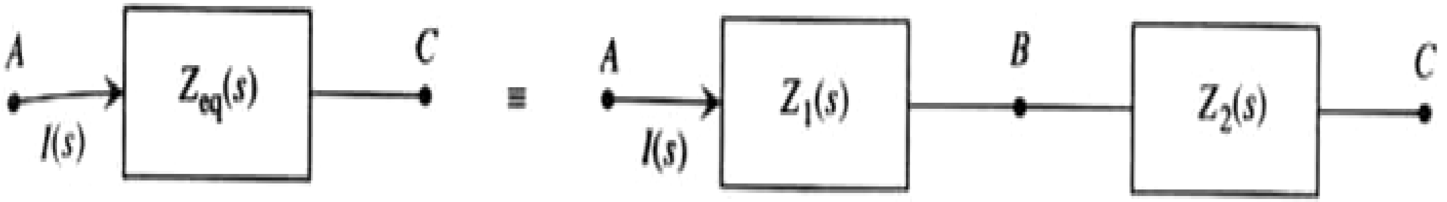
For the inductor,

$$V(s) = LsI(s)$$

Elements of Electrical System: Impedance Method

Theorem 1. If there are *n* impedance in **series**

$$Z_{eq}(s) = Z_1(s) + Z_2(s) + \cdots + Z_n(s)$$



Equivalent impedance for two impedances in series

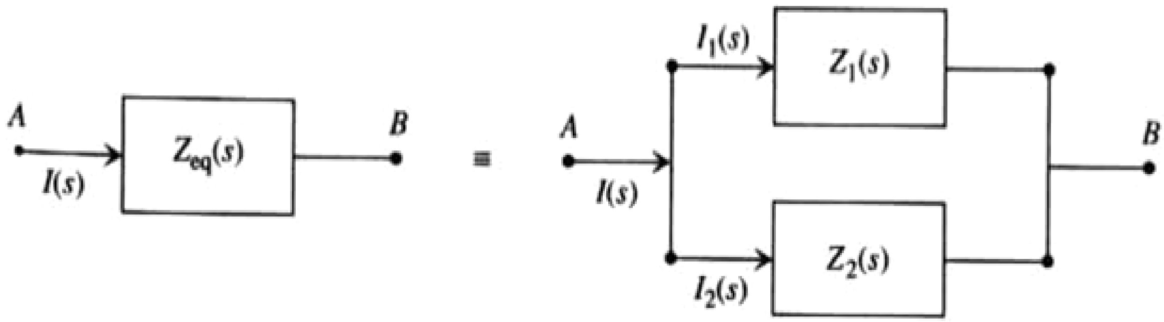
$$Z_{eq}(s) = Z_1(s) + Z_2(s)$$

Elements of Electrical System

Impedance Method

Theorem 2. If there are *n* impedance in **parallel**

$$\frac{1}{Z_{eq}(s)}=\frac{1}{Z_1(s)}+\frac{1}{Z_2(s)}+\cdots+\frac{1}{Z_n(s)}$$



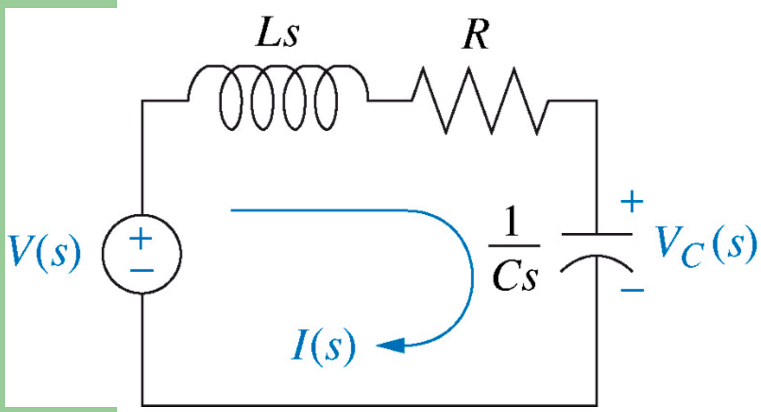
Equivalent impedance for two impedances in parallel

$$\frac{1}{Z_{eq}(s)}=\frac{1}{Z_1(s)}+\frac{1}{Z_2(s)}$$

ELECTRICAL NETWORK TRANSFER FUNCTIONS

Impedance Method

$$\frac{V(s)}{I(s)} = Z(s)$$



$$\sum v_{drop} = 0$$

$$v_L + v_R + v_C - v = 0$$

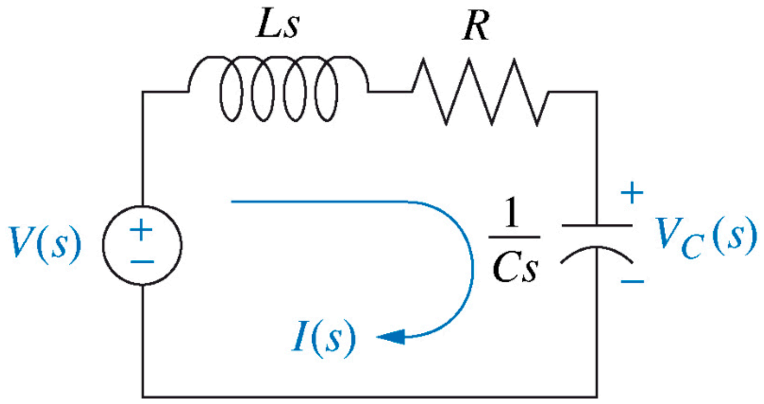
$$V_L(s) + V_R(s) + V_C(s) - V(s) = 0$$

$$LsI(s) + RI(s) + V_C(s) - V(s) = 0$$

$$Ls(CsV_C(s)) + R(CsV_C(s)) + V_C(s) - V(s) = 0$$

$$\left(LCs^2 + RCs + 1\right)V_C(s) = V(s)$$

ELECTRICAL NETWORK TRANSFER FUNCTIONS

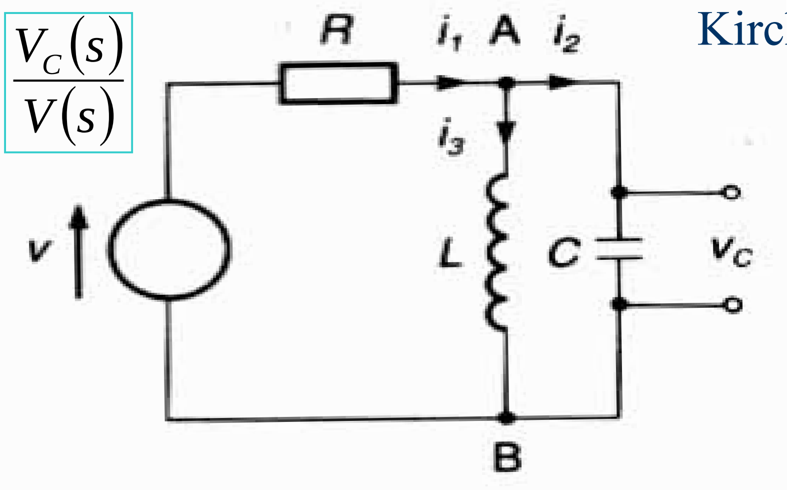


$$\left(LCs^2 + RCs + 1\right)V_C(s) = V(s)$$

$$\frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1}$$

ELECTRICAL NETWORK TRANSFER FUNCTIONS

$$\frac{V_C(s)}{V(s)}$$



Kirchhoff's Current Law (KCL), Node method

$$\sum_k (i_k)_{in} = 0$$

$$i_1 - i_2 - i_3 = 0 \Leftrightarrow i_2 + i_3 = i_1$$

ELECTRICAL NETWORK TRANSFER FUNCTIONS

$$i_1 = \frac{v - v_A}{R}, i_2 = C\frac{dv_A}{dt}, \ i_3 = \frac{1}{L}\int v_A dt$$

$$i_1 = i_2 + i_3$$

$$\frac{v - v_A}{R} = C\frac{dv_A}{dt} + \frac{1}{L}\int v_A dt$$

$$v_A = v_C$$

$$v = RC\frac{dv_C}{dt} + v_C + \frac{R}{L}\int v_C dt$$

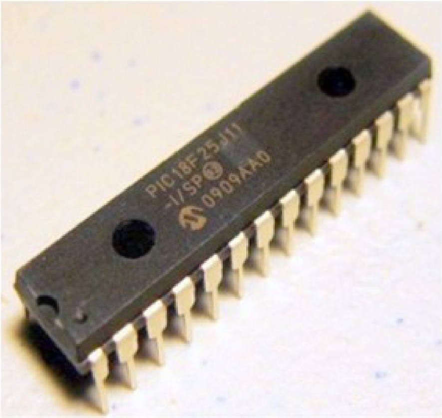
ELECTRICAL NETWORK TRANSFER FUNCTIONS

$$V(s) = RCsV_C(s) + V_C(s) + \frac{R}{L}\frac{1}{s}V_C(s)$$

Transfer function

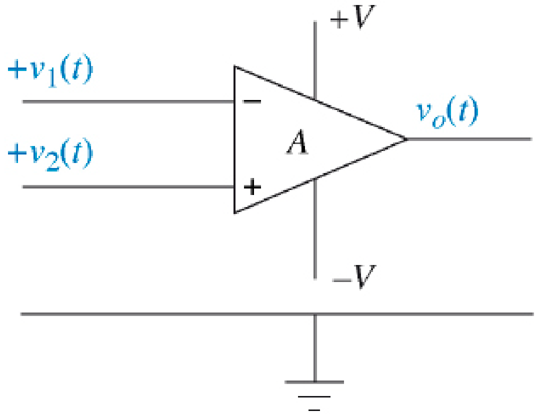
$$\frac{V_C(s)}{V(s)} = \frac{Ls}{LRCs^2 + Ls + R}$$

OPERATIONAL AMPLIFIERS

Description	PIC18F25J11	PIC24F16KA102
Packaging	28 Pin PDIP 	28 Pin PDIP 
Register	8 Bit	16 Bit
Program Memory	16 KByte	16 KByte
Instruction Clock	4 Cycles	2 Cycles
Price	USD 2.70	USD 2.41
Development Tools	Microchip MPLAB and Microchip C18 or Microchip HI-TECH C for PIC18 C Compiler	Microchip MPLAB and Microchip C30, Microchip HI-TECH C for dsPIC/PIC24 C Compiler
Peripherals	Advance	More Advance
Speed	Up to 12 MIPS at 48 MHz	Up to 16 MIPS at 32 MHz

Microchip PIC18F25J11 and PIC24F16KA102 Microcontroler Basic Comparison

OPERATIONAL AMPLIFIERS

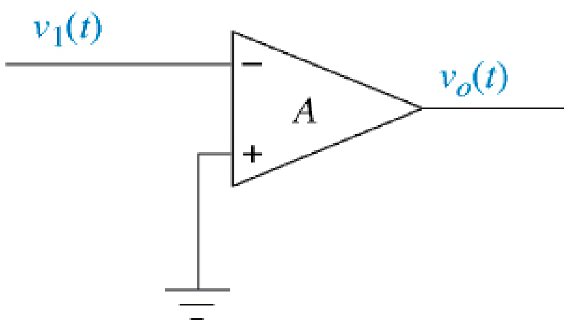


(a)

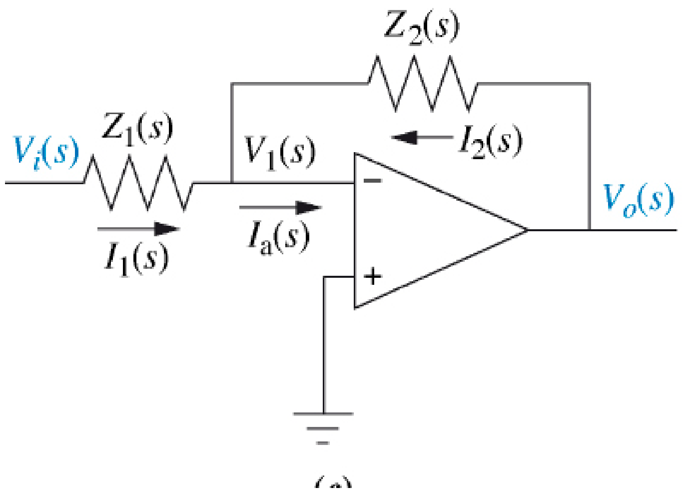
- Inverting Op-Amp
- Non-inverting Op-Amp

1. Differential input, $v_2(t) - v_1(t)$
2. High input impedance, $Z_i = \infty$ (ideal)
3. Low output impedance, $Z_o = 0$ (ideal)
4. High constant gain amplification, $A = \infty$ (ideal)

INVERTING OPERATIONAL AMPLIFIERS



$$v_0(t) = -Av_1(t)$$



$$I_a(s) = 0$$

$$I_1(s) = -I_2(s)$$

INVERTING OPERATIONAL AMPLIFIERS

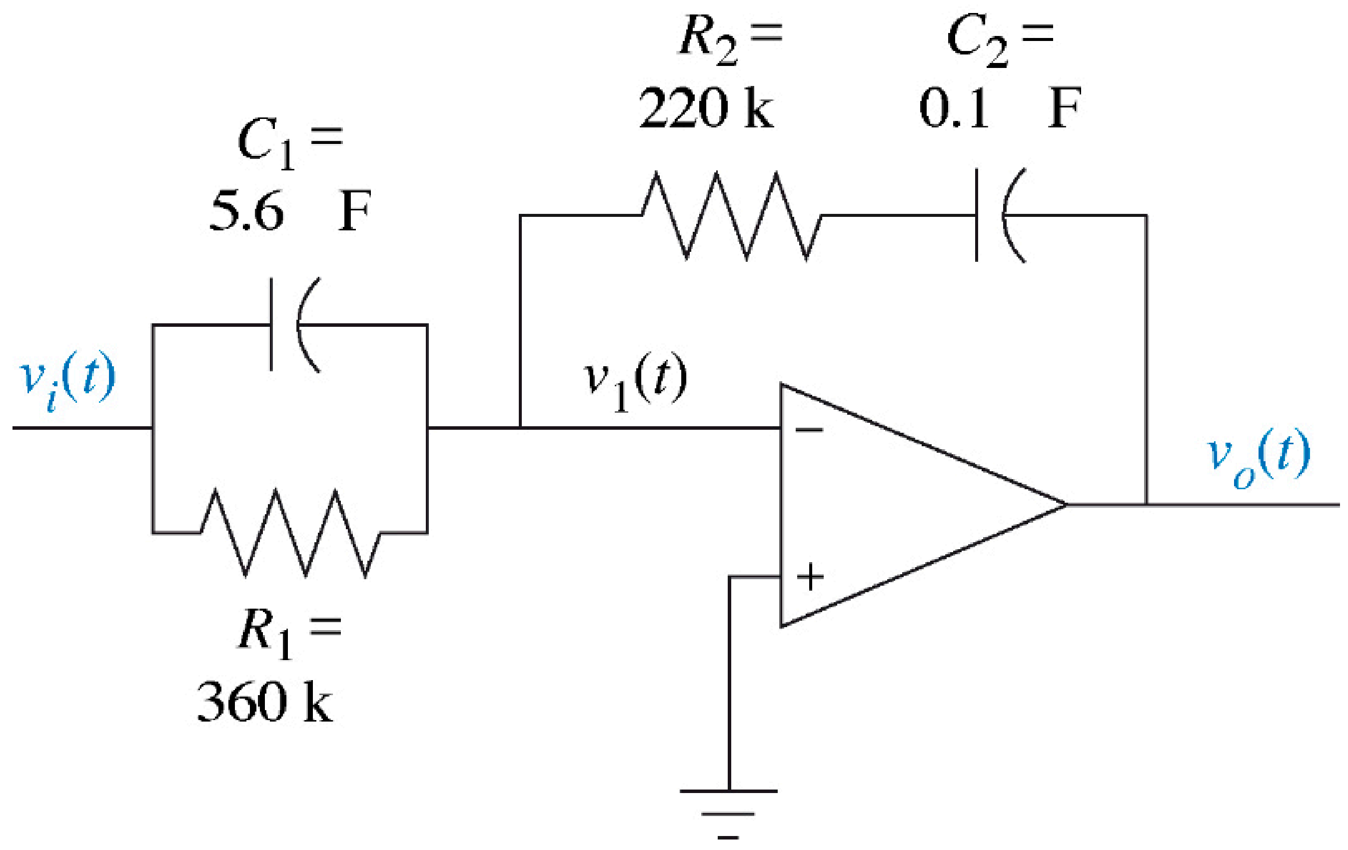
$$I_1(s) = \frac{V_i(s)}{Z_1(s)}$$

$$I_2(s) = \frac{V_0(s)}{Z_2(s)}$$

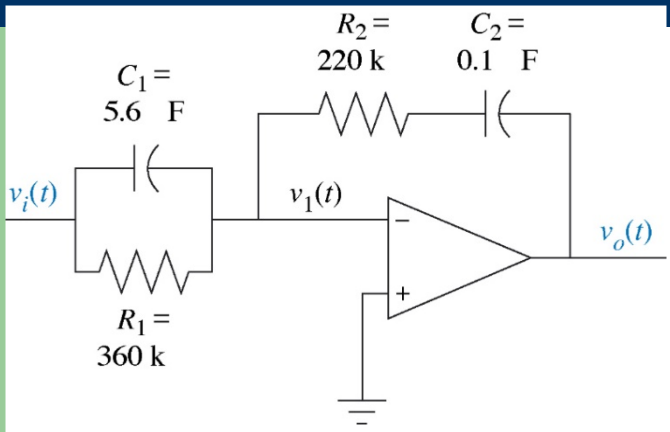
Transfer function

$$\frac{V_0(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

INVERTING OPERATIONAL AMPLIFIERS: PID Controller



INVERTING OPERATIONAL AMPLIFIERS: PID Controller

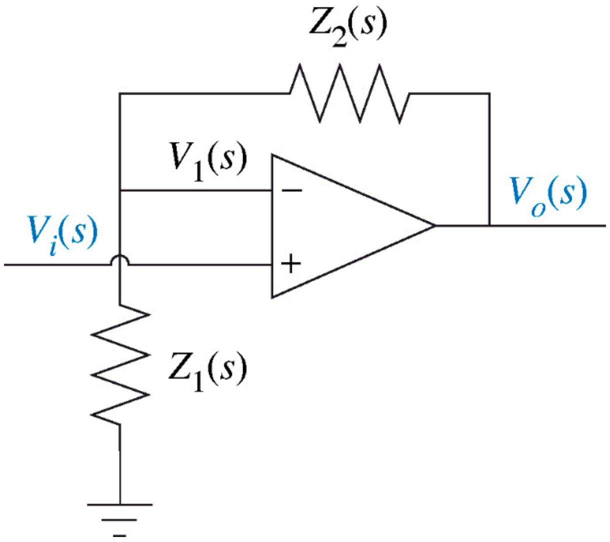


$$\frac{V_o(s)}{V_i(s)} = -1.232\frac{s^2 + 45.95s + 22.55}{s}$$

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6}s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016s + 1} \quad (2.98)$$

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s} \quad (2.99)$$

NON-INVERTING OPERATIONAL AMPLIFIERS



$$V_o(s) = A(V_i(s) - V_1(s))$$

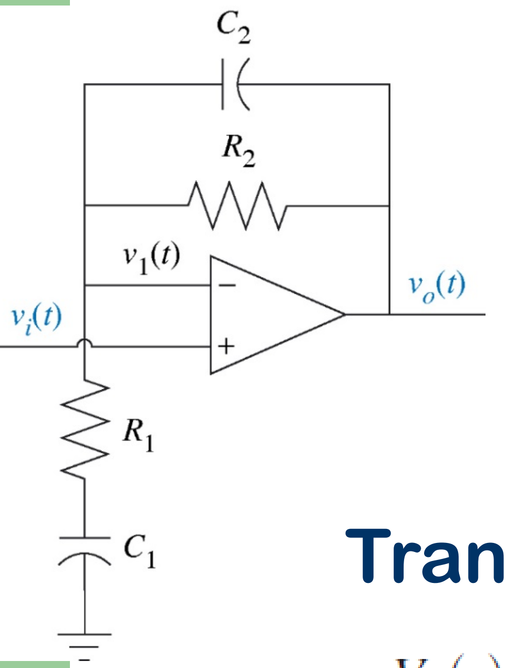
$$V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s) + Z_2(s))}$$

For large A, we disregard unity in the denominator

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$

NON-INVERTING OPERATIONAL AMPLIFIERS



$$Z_1(s) = R_1 + \frac{1}{C_1 s}$$

$$Z_2(s) = \frac{R_2(1/C_2 s)}{R_2 + (1/C_2 s)}$$

Transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_2 + C_1R_1)s + 1}{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_1)s + 1}$$