Presenting Frequency-Response

Presenting frequency-response characteristics in graphical forms

- Bode diagram or logarithm plot
- Nyquist plot or polar plot
- Log-magnitude-versus-phase plot (Nichols plots)

Bode diagrams

- Gain, K
- Integral and derivative factors
- First-order factors
- Quadratic factors

The gain, *K*

Case 1 $G(j\omega) = K$

The logarithmic magnitude of $G(j\omega)$ in decibels is

$$20\log\left|G(j\omega)\right| = 20\log K = \text{constant}$$

The phase angle of $G(j\omega)$ is

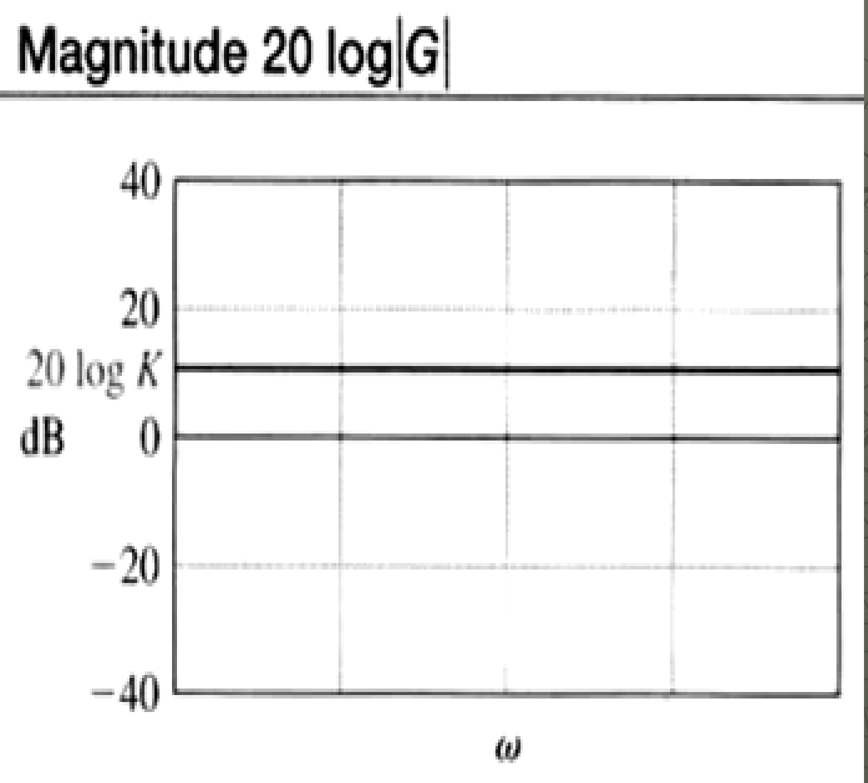
$$\phi = \tan^{-1}\frac{\text{Im}}{\text{Re}} = 0^\circ$$

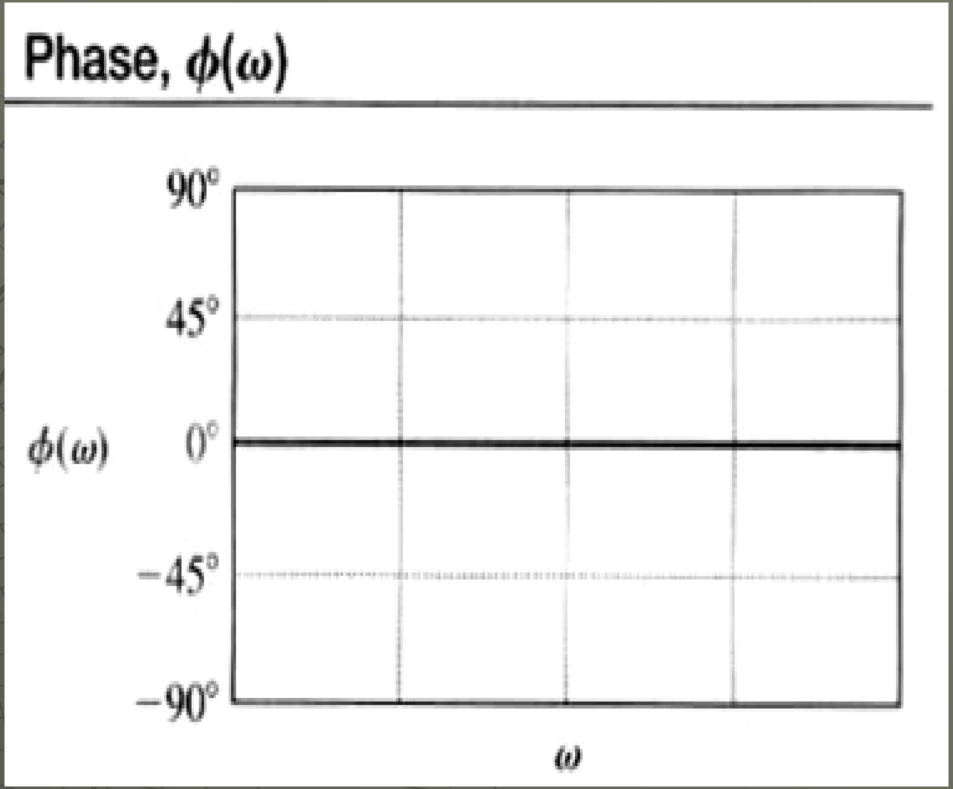
where

$\text{Re}(\omega) = K; \text{Im}(\omega) = 0$

$$\left|G(j\omega)\right| = \sqrt{(\text{Re})^2 + (\text{Im})^2} = K$$

The gain, *K*


Magnitude 20 log|G|
40
20
20 log K
dB
0
−20
−40
ω


Phase, ϕ(ω)
90°
45°
ϕ(ω)
0°
−45°
−90°
ω

Integral Factor

Case 2 $$G(j\omega) = \frac{1}{j\omega}$$

The logarithmic magnitude of $G(j\omega)$ in decibels is

$$20\log\left|\frac{1}{j\omega}\right| = -20\log\omega \quad \text{dB}$$

The phase angle of $G(j\omega)$ is

$$\phi = \tan^{-1}\frac{\mathrm{Im}}{\mathrm{Re}} = -90^\circ$$

where

$$\mathrm{Re}(\omega) = 0; \mathrm{Im}(\omega) = -1/\omega$$

$$|G(j\omega)| = \sqrt{(-1/\omega)^2} = \frac{1}{\omega}$$

Integral Factor

From

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\text{Im}}{\text{Re}} = -\frac{1/\omega}{0}$$

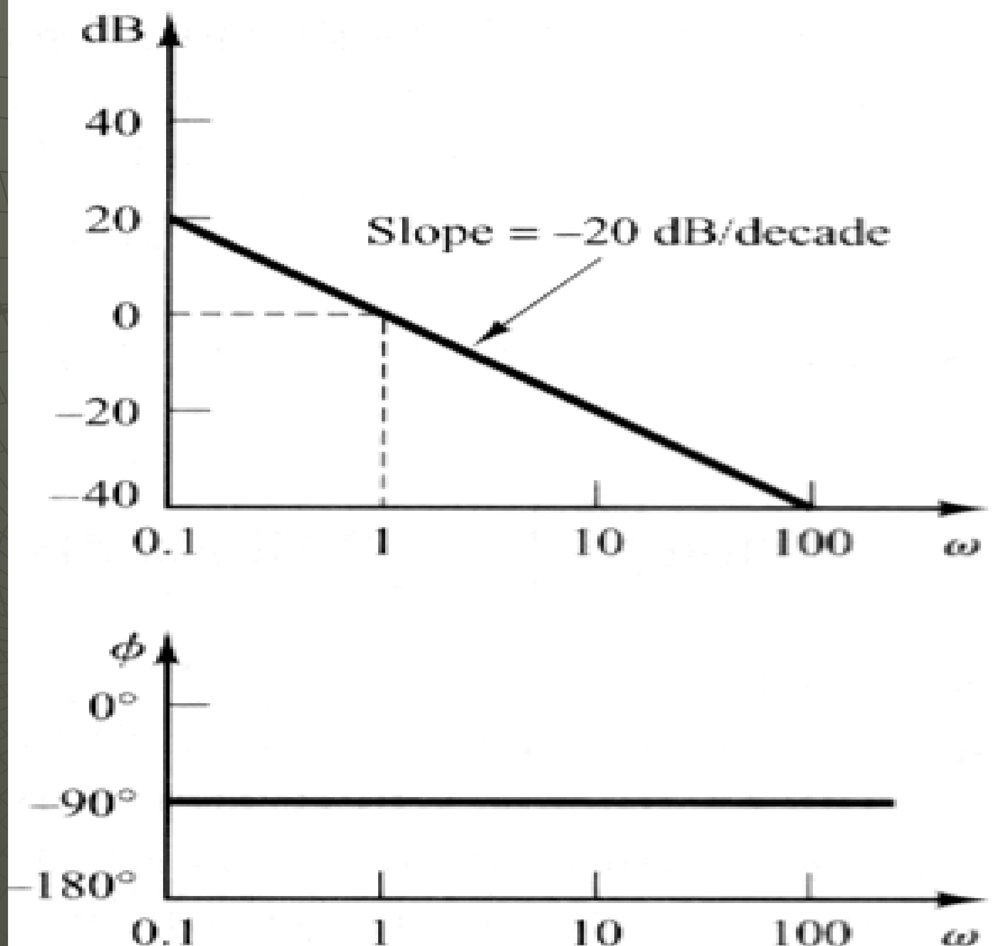
$$\text{Re}\cdot\sin\phi = \text{Im}\cdot\cos\phi$$

$$0 = \text{Im}\cdot\cos\phi \rightarrow 0 = -1/\omega\cdot\cos\phi$$

$$-\phi = \cos^{-1}\frac{0}{1/\omega} = 90^\circ$$

$$\phi = -90^\circ$$

Integral Factor



Bode diagram of
$G(j\omega) = 1/j\omega$

Derivative Factor

Case 3 $G(j\omega) = j\omega$

The logarithmic magnitude of $G(j\omega)$ in decibels is

$$20\log|j\omega| = 20\log\omega \quad \text{dB}$$

The phase angle of $G(j\omega)$ is

$$\phi = \tan^{-1}\frac{\text{Im}}{\text{Re}} = 90^\circ$$

where

$$\text{Re}(\omega) = 0; \text{Im}(\omega) = \omega$$

$$|G(j\omega)| = \sqrt{(\omega)^2} = \omega$$

Derivative Factor

From

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\mathrm{Im}}{\mathrm{Re}}$$

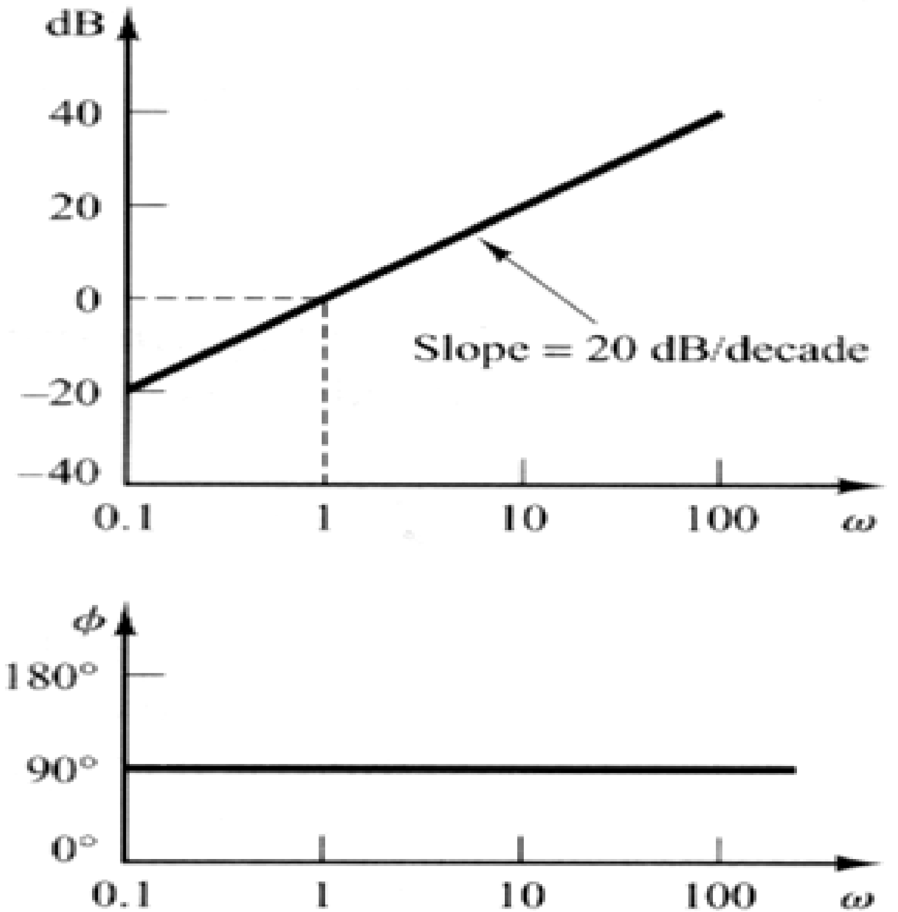
$$\mathrm{Re}\cdot\sin\phi = \mathrm{Im}\cdot\cos\phi$$

$$0 = \mathrm{Im}\cdot\cos\phi \rightarrow 0 = \omega\cdot\cos\phi$$

$$\phi = \cos^{-1}\frac{0}{\omega} = 90^\circ$$

$$\phi = 90^\circ$$

Derivative Factor



Bode diagram of
$G(j\omega) = j\omega$

First Order Factors
real pole

Case 4 $$G(j\omega)=\frac{1}{1+j\omega T}$$

The logarithmic magnitude of $G(j\omega)$ in decibels is

$$20\log\left|\frac{1}{1+j\omega T}\right|=-20\log\sqrt{1+\omega^2T^2}\quad \text{dB}$$

where $$\text{Re}(\omega)=\frac{1}{1+\omega^2T^2};\text{Im}(\omega)=-\frac{\omega T}{1+\omega^2T^2}$$

$$|G(j\omega)|=\sqrt{\left(\frac{1}{1+\omega^2T^2}\right)^2+\left(\frac{\omega T}{1+\omega^2T^2}\right)}=\frac{1}{\sqrt{1+\omega^2T^2}}$$

First Order Factors
real pole

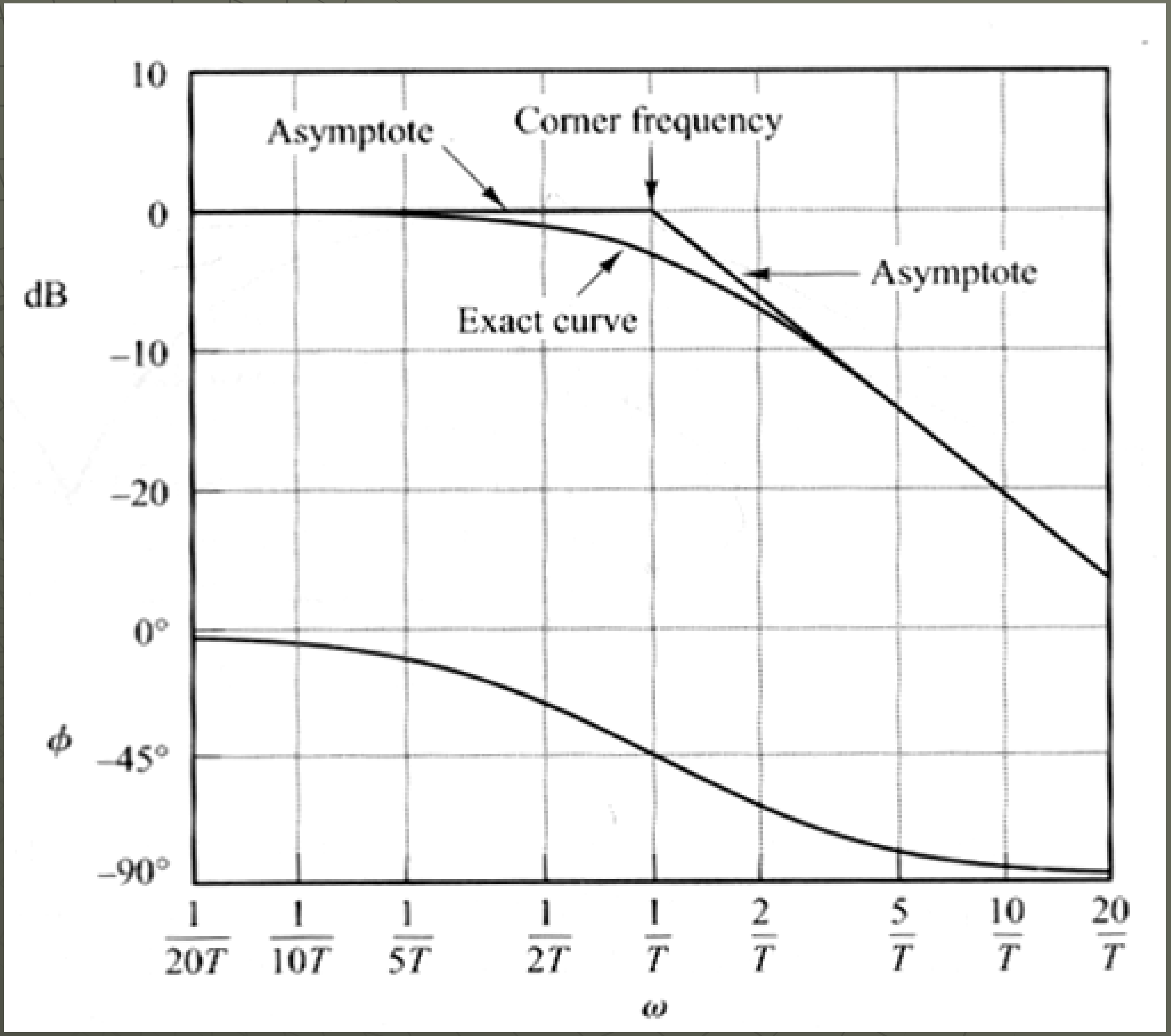
where

$$\phi = \tan^{-1}\frac{\mathrm{Im}}{\mathrm{Re}} = -\tan^{-1}\omega T$$

From

$$\tan\phi = \frac{\mathrm{Im}}{\mathrm{Re}} = \frac{-\omega T/1+\omega^2 T^2}{1/1+\omega^2 T^2} = -\omega T$$

First Order Factors
real pole



First Order Factors
real zero

Case 5 $G(j\omega) = 1 + j\omega T$

The logarithmic magnitude of $G(j\omega)$ in decibels is

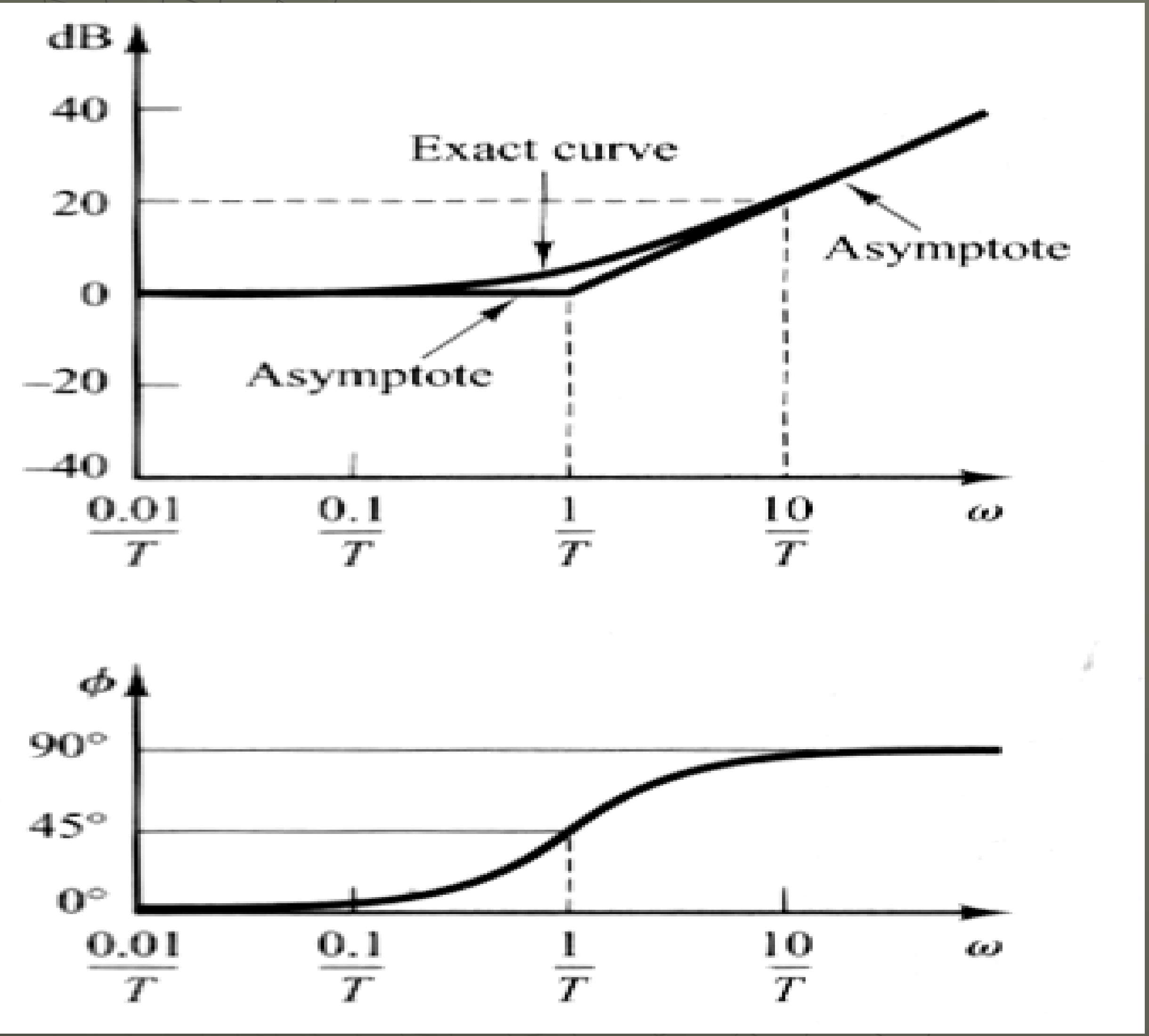
$$20\log|1 + j\omega T| = 20\log\sqrt{1+\omega^2 T^2} \quad \text{dB}$$

The phase angle of $G(j\omega)$ is

$$\phi = \tan^{-1}\frac{\text{Im}}{\text{Re}} = \tan^{-1}\omega T$$

where $\text{Re}(\omega) = 1; \text{Im}(\omega) = \omega T$ $\quad |G(j\omega)| = \sqrt{1+\omega^2 T^2}$

First Order Factors
real zero



Second Order Factors
complex poles

Case 6

$$G(s) = \frac{\omega_n^2}{\omega_n^2 + 2\zeta\omega_n s + s^2}$$

or

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\dfrac{\omega}{\omega_n}\right) + \left(j\dfrac{\omega}{\omega_n}\right)^2}$$

or

$$G(j\omega) = \frac{1}{1 + j2\zeta u + (ju)^2}; \quad u = \frac{\omega}{\omega_n}$$

Second Order Factors
complex poles

The logarithmic magnitude of $G(j\omega)$ in decibels is

$$20\log\left|\frac{1}{1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2}\right| = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

or

$$20\log\left|\frac{1}{1+j2\zeta u+(ju)^2}\right| = -20\log\sqrt{\left(1-u^2\right)^2+\left(2\zeta u\right)^2}$$

Second Order Factors
complex poles

Conjugate

$$G(j\omega)=\frac{1}{1+j2\zeta u+(ju)^2}=\frac{1}{\left(1-u^2\right)+j2\zeta u}$$

$$=\frac{1}{\left(1-u^2\right)+j2\zeta u}\cdot\frac{\left(1-u^2\right)-j2\zeta u}{\left(1-u^2\right)-j2\zeta u}$$

$$=\frac{1-u^2}{\left(1-u^2\right)^2+(2\zeta u)^2}-j\frac{2\zeta u}{\left(1-u^2\right)^2+(2\zeta u)^2}$$

where

$$\operatorname{Im}(\omega)=\frac{2\zeta u}{\left(1-u^2\right)^2+(2\zeta u)^2}$$

$$\operatorname{Re}(\omega)=\frac{1-u^2}{\left(1-u^2\right)^2+(2\zeta u)^2}$$

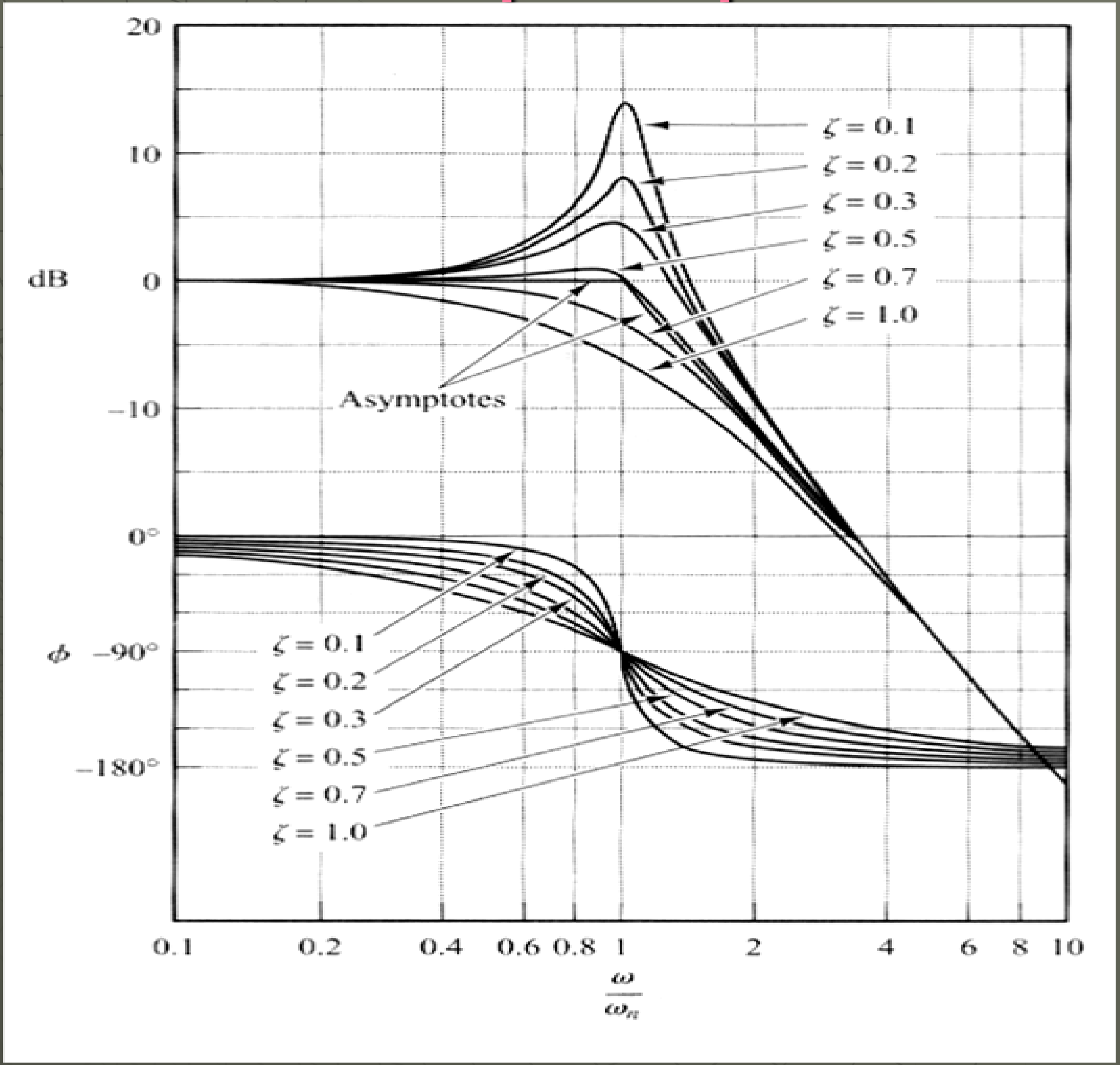
Second Order Factors
complex poles

The phase angle of $G(j\omega)$ is

$$\phi = \tan^{-1}\frac{\text{Im}}{\text{Re}}$$

$$= -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right) = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right)$$

Second Order Factors
complex pole



Second Order Factors
complex pole

The Resonant Frequency ω_r and the Resonant Peak Value M_r. The magnitude of

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

is

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \tag{8–9}$$

If $|G(j\omega)|$ has a peak value at some frequency, this frequency is called the *resonant* frequency. Since the numerator of $|G(j\omega)|$ is constant, a peak value of $|G(j\omega)|$ will occur when

$$g(\omega) = \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2 \tag{8–10}$$

is a minimum. Since Equation (8–10) can be written

$$g(\omega) = \left[\frac{\omega^2 - \omega_n^2(1 - 2\zeta^2)}{\omega_n^2}\right]^2 + 4\zeta^2(1 - \zeta^2) \tag{8–11}$$

the minimum value of $g(\omega)$ occurs at $\omega = \omega_n\sqrt{1 - 2\zeta^2}$. Thus the resonant frequency ω_r is

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}, \qquad \text{for } 0 \le \zeta \le 0.707 \tag{8–12}$$

As the damping ratio ζ approaches zero, the resonant frequency approaches ω_n. For $0 < \zeta \le 0.707$, the resonant frequency ω_r is less than the damped natural frequency $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, which is exhibited in the transient response. From Equation (8–12), it can be seen that for $\zeta > 0.707$, there is no resonant peak. The magnitude $|G(j\omega)|$ decreases monotonically with increasing frequency ω. (The magnitude is less than 0 dB for all values of $\omega > 0$. Recall that, for $0.7 < \zeta < 1$, the step response is oscillatory, but the oscillations are well damped and are hardly perceptible.)

Second Order Factors
complex pole

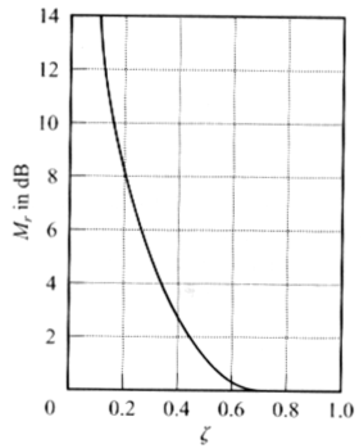


Figure 8–10
M_r-versus-ζ curve for the second-order system $1/[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2]$.

The magnitude of the resonant peak, M_r, can be found by substituting Equation (8–12) into Equation (8–9). For $0 \le \zeta \le 0.707$,

$$M_r = |G(j\omega)|_{\max} = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{8–13}$$

For $\zeta > 0.707$,

$$M_r = 1 \tag{8–14}$$

As ζ approaches zero, M_r approaches infinity. This means that if the undamped system is excited at its natural frequency, the magnitude of $G(j\omega)$ becomes infinity. The relationship between M_r and ζ is shown in Figure 8–10.

The phase angle of $G(j\omega)$ at the frequency where the resonant peak occurs can be obtained by substituting Equation (8–12) into Equation (8–8). Thus, at the resonant frequency ω_r,

$$\underline{/G(j\omega_r)} = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta} = -90° + \sin^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

Ex: Plotting Bode diagram

Consider the following transfer function:

$$G(s)=\frac{10}{s(0.1s+1)} \quad \Rightarrow \quad G(j\omega)=\frac{10}{j\omega(0.1j\omega+1)}$$

The log magnitude is

$$20\log|G(j\omega)| = 20\log\left|\frac{10}{j\omega(0.1j\omega+1)}\right|$$
$$= 20\log|10| + 20\log\left|\frac{1}{j\omega}\right| + 20\log\left|\frac{1}{j0.1\omega+1}\right|$$

Ex: Plotting Bode diagram

Term 1:

$$20\log|10| = 20\log 10 = 20 \quad \text{dB}$$

Term 2:

$$20\log\left|\frac{1}{j\omega}\right| = -20\log|j\omega| = -20\log\omega \quad \text{dB}$$

ω rad/s	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
Y, dB	40	20	0	-20	-40	-60

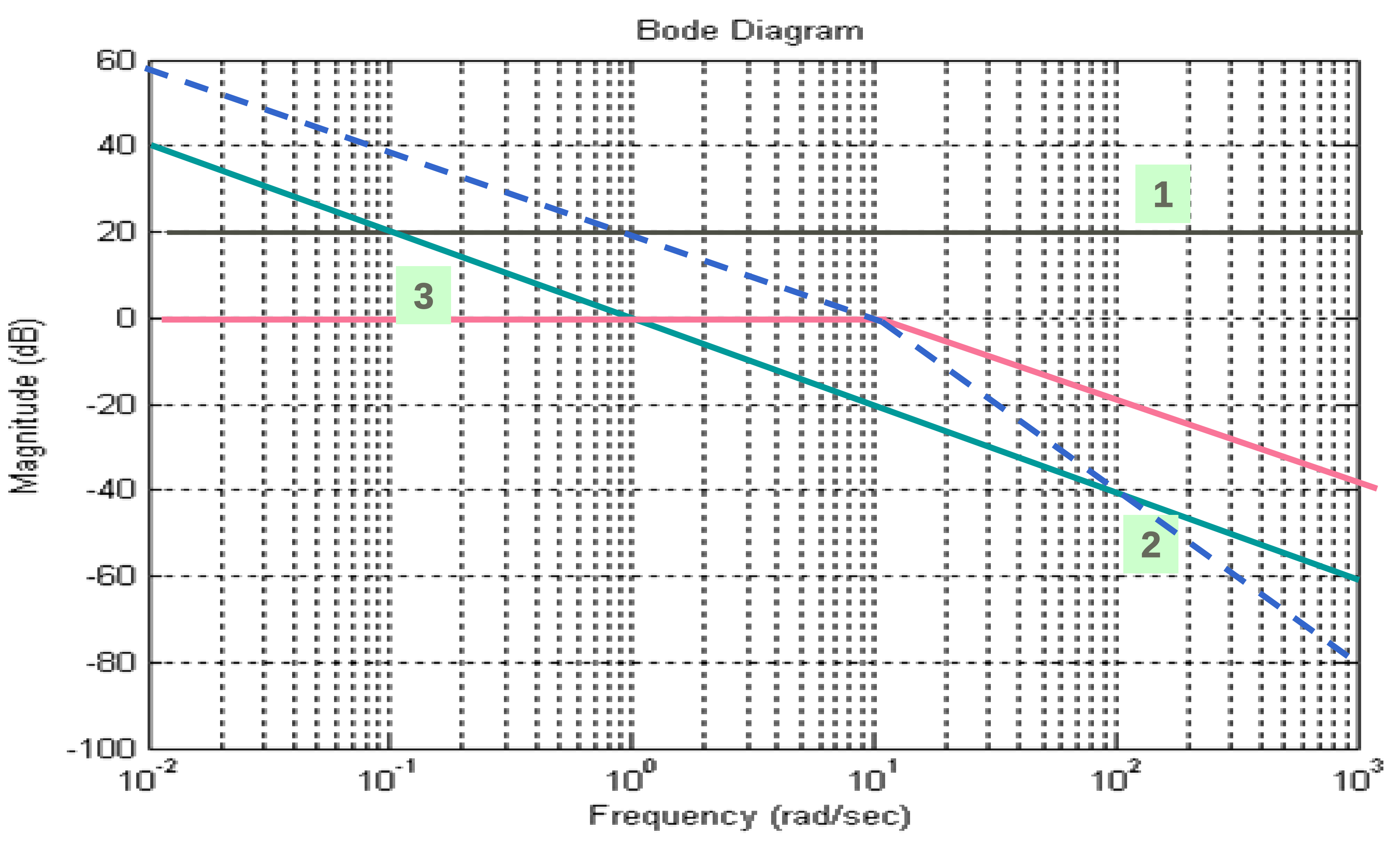
Ex: Plotting Bode diagram

Term 3:

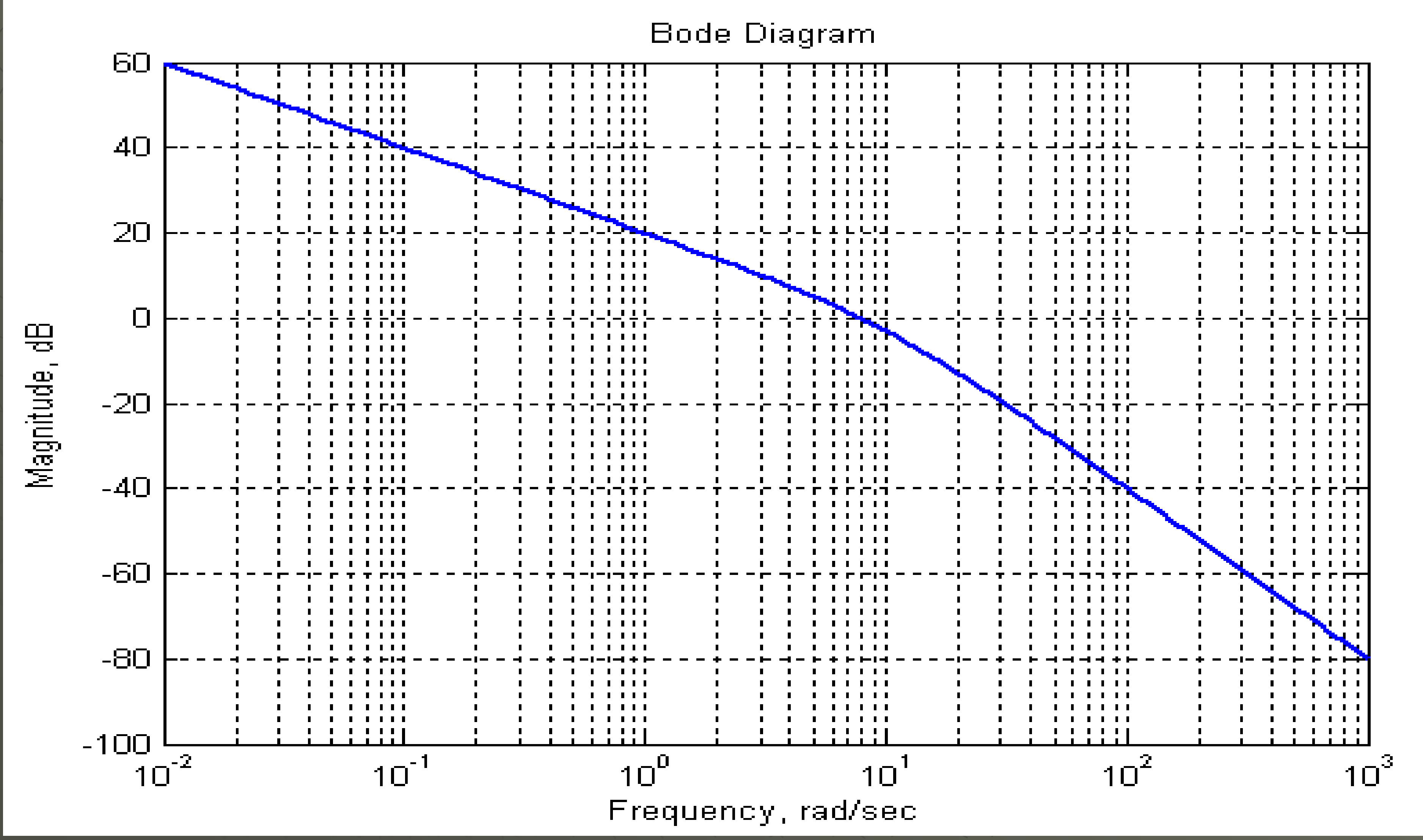
$$20\log\left|\frac{1}{j0.1\omega+1}\right| = -20\log\left(1+(0.1\omega)^2\right)^{1/2}$$
$$= -10\log\left(1+(0.1\omega)^2\right)$$

ω rad/s	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
Y,dB	0	0	0	0	-20	-40

Bode Diagram
asymptote magnitude plot



Bode Diagram
exact magnitude curve



Ex: Plotting Bode diagram

The phase angle is

Term 1: $G(j\omega) = K = 10$

$$\phi = \tan^{-1}\frac{\text{Im}}{\text{Re}} = 0^{\circ}$$

Term 2: $$G(j\omega) = \frac{1}{j\omega}$$

$$\phi = \tan^{-1}\frac{\text{Im}}{\text{Re}} = -90^{\circ}$$

Ex: Plotting Bode diagram

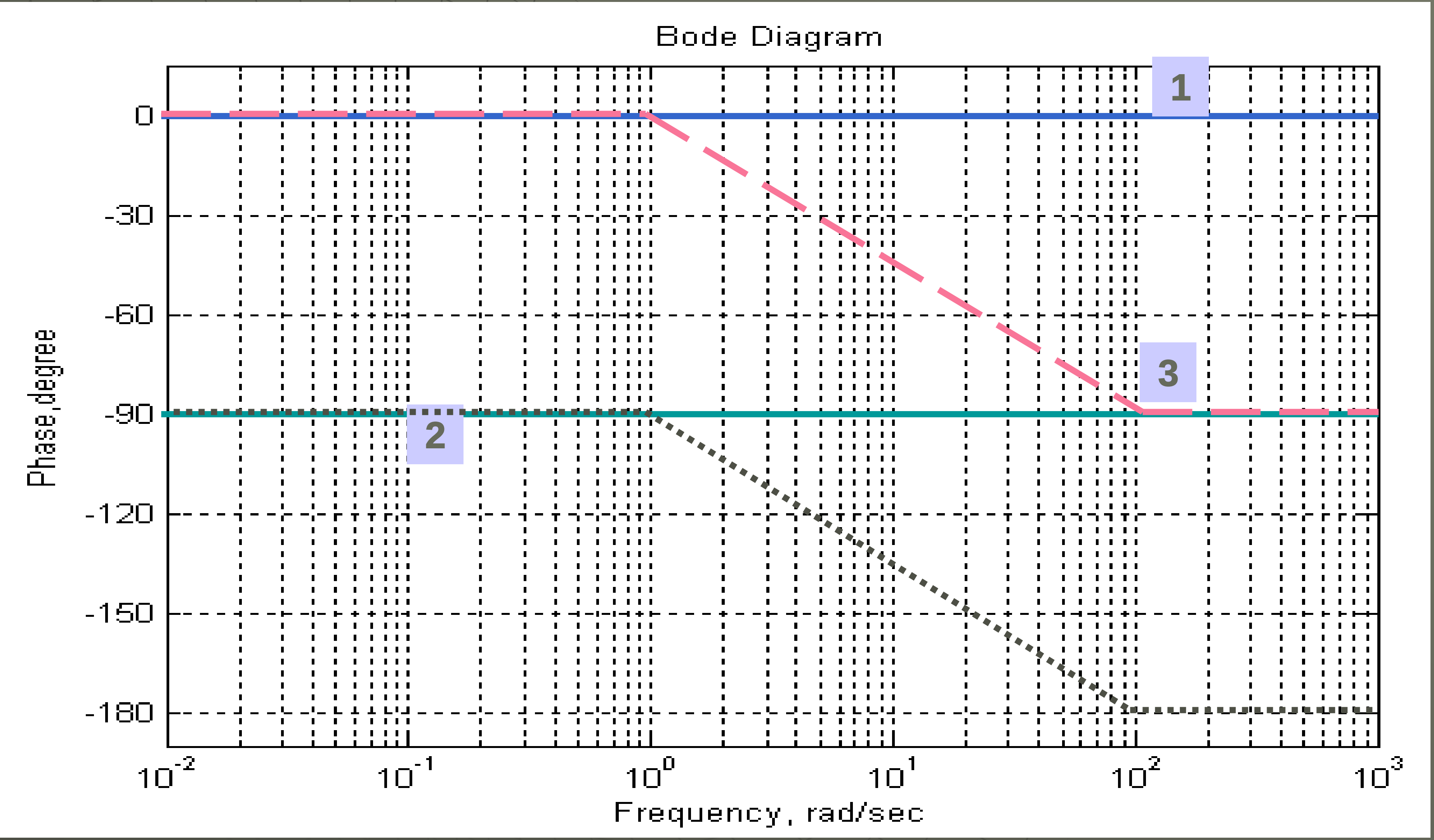
Term 3:

$$G(j\omega) = \frac{1}{j0.1\omega + 1}$$

$$\phi = \tan^{-1}\frac{\text{Im}}{\text{Re}} = -\tan^{-1} 0.1\omega$$

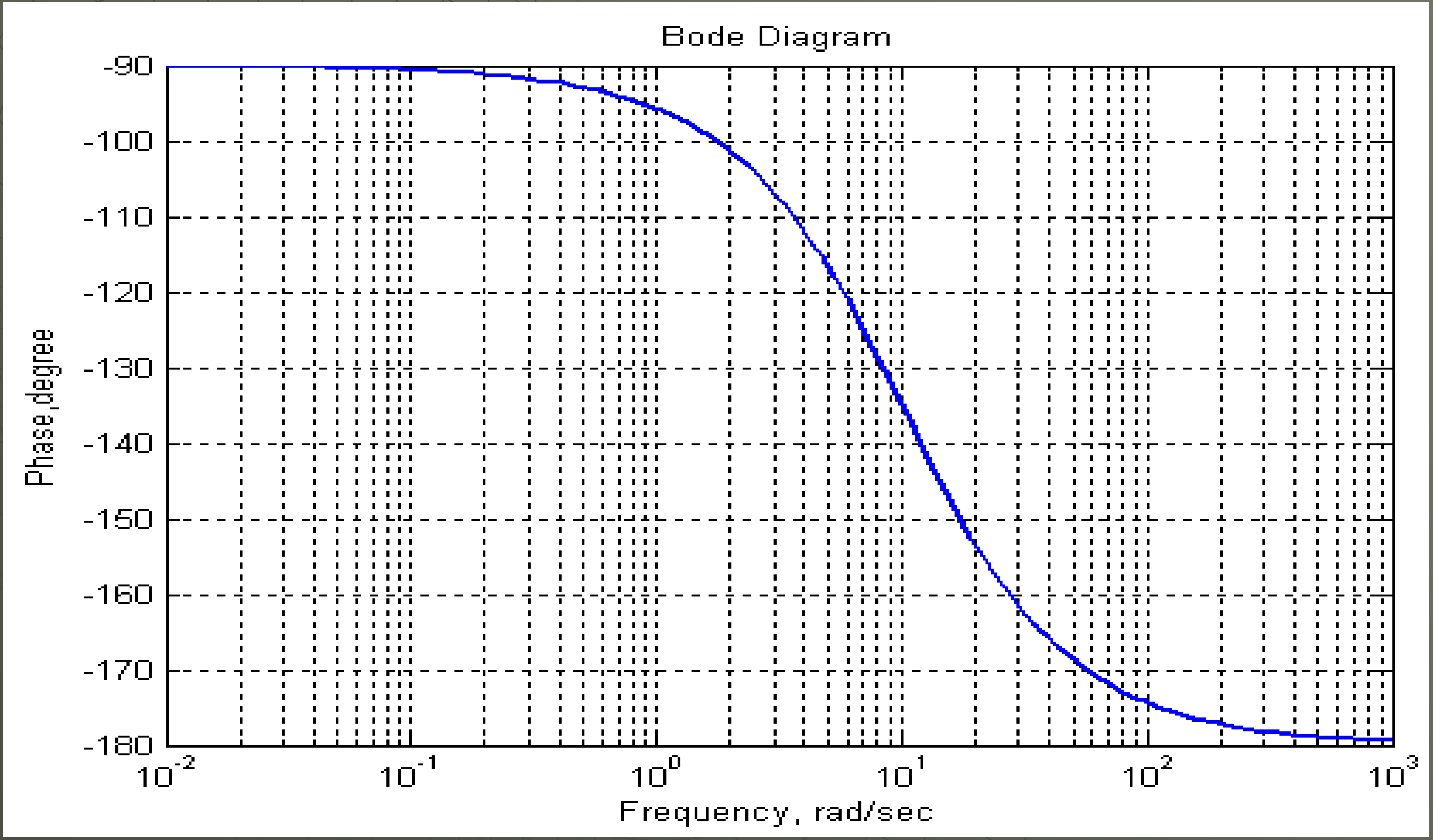
ω rad/s	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
P degree	0	0	0	-45	-90	-90

Bode Diagram
asymptote phase plot



Bode Diagram

exact phase curve



ExII: Plotting Bode diagram

Consider the following transfer function:

$$G(s)=\frac{10(s+3)}{(s^2+s+2)} \quad\Rightarrow\quad G(j\omega)=\frac{10\cdot 3\left(\frac{j\omega}{3}+1\right)}{2\left(\frac{(j\omega)^2}{2}+\frac{1}{2}j\omega+1\right)}$$

$$G(j\omega)=\frac{15\left(j\frac{\omega}{3}+1\right)}{\left(\frac{(j\omega)^2}{2}+j\frac{\omega}{2}+1\right)}$$

ExII: Plotting Bode diagram

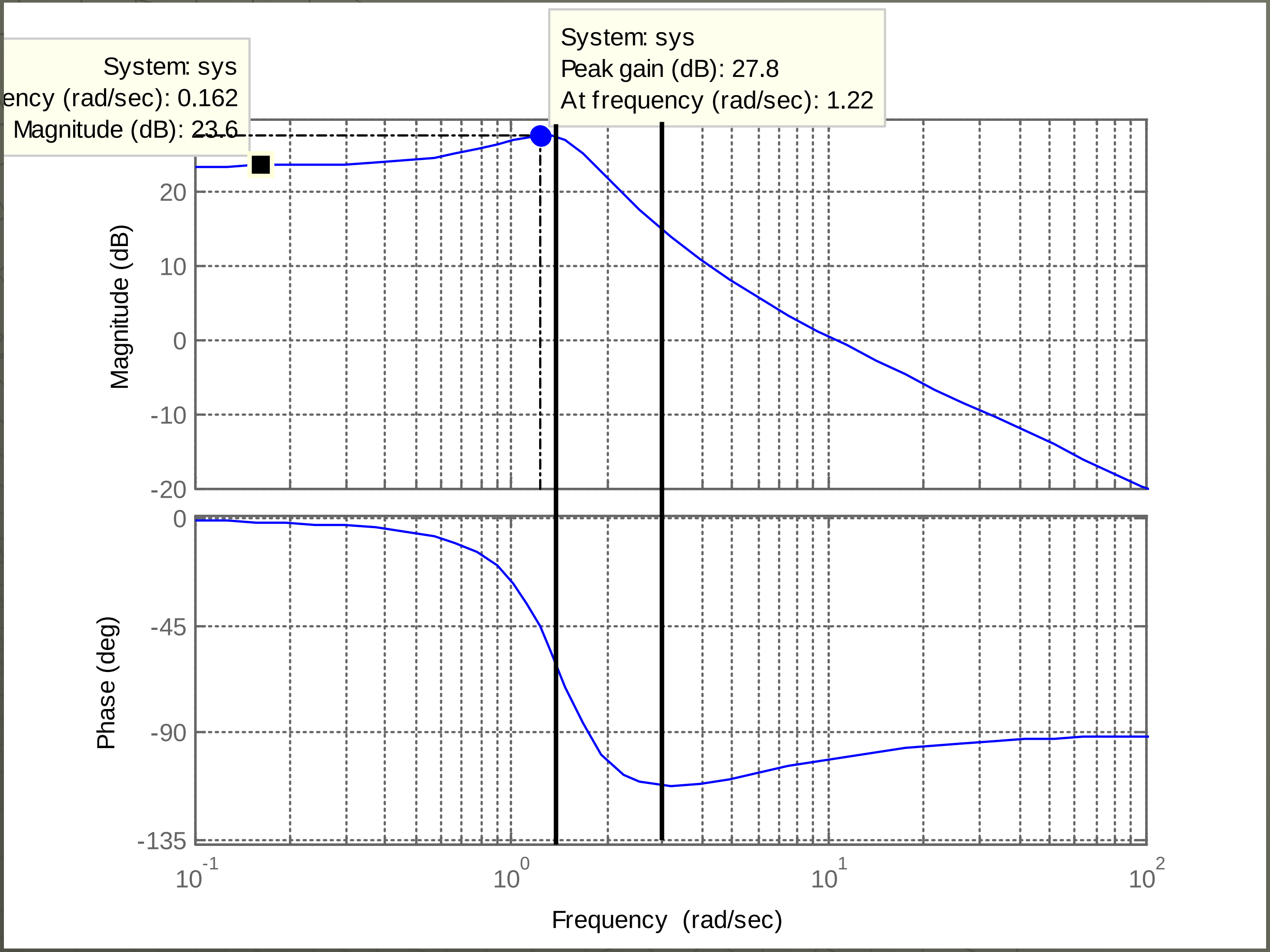
The log magnitude is

$$20\log\left|G(j\omega)\right| = 20\log\left|\frac{15(j0.33\omega+1)}{0.5(j\omega)^2 + j0.5\omega+1}\right|$$

$$= 20\log|15| + 20\log\left|j0.33\omega+1\right| + 20\log\left|\frac{1}{0.5(j\omega)^2 + j0.5\omega+1}\right|$$

$$= 20\log|15| + 20\log\left|j\frac{\omega}{3}+1\right| + 20\log\left|\frac{1}{\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\frac{1}{\sqrt{2}}\cdot\frac{\omega}{\sqrt{2}}+1}\right|$$

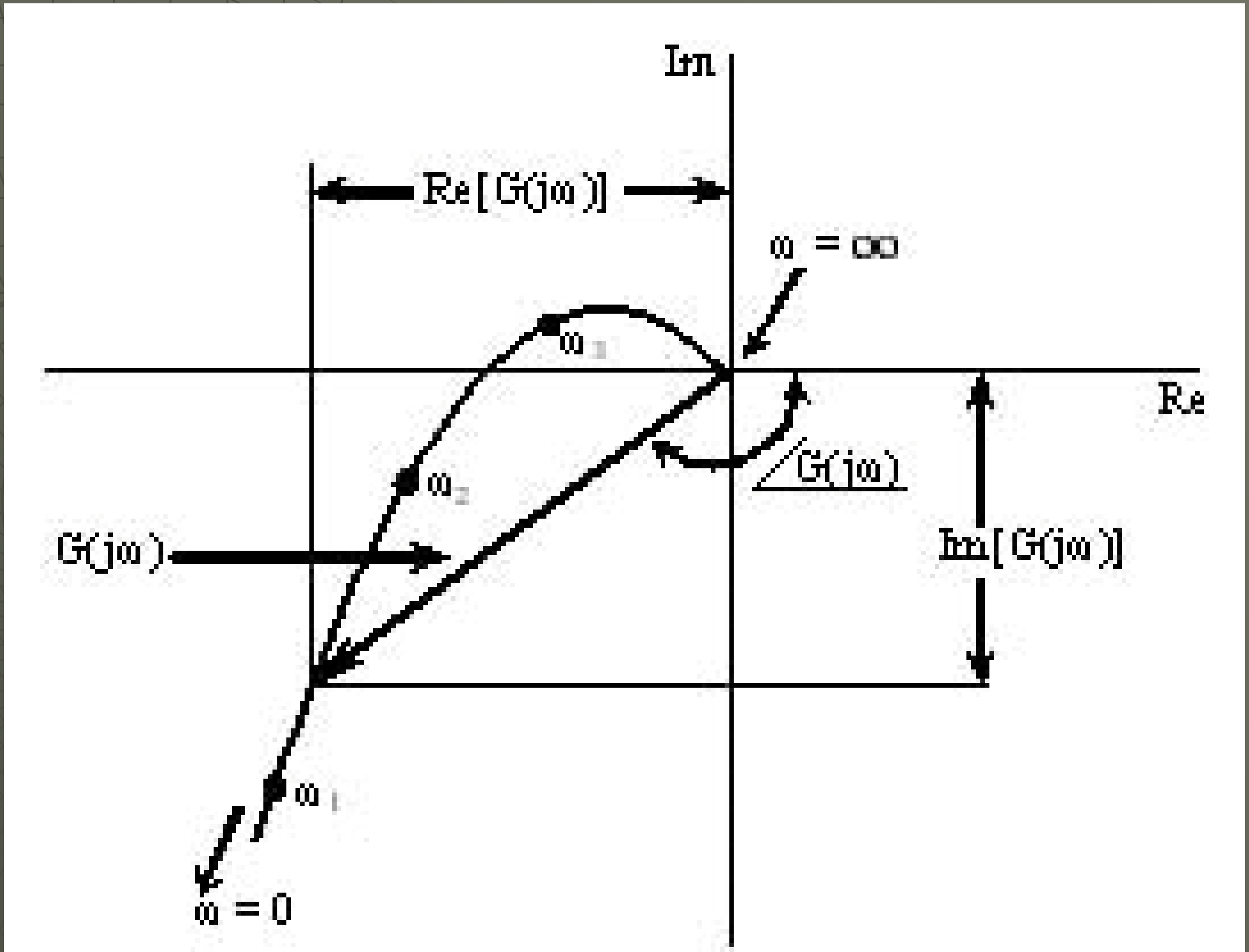
Ex-II: Plotting Bode diagram



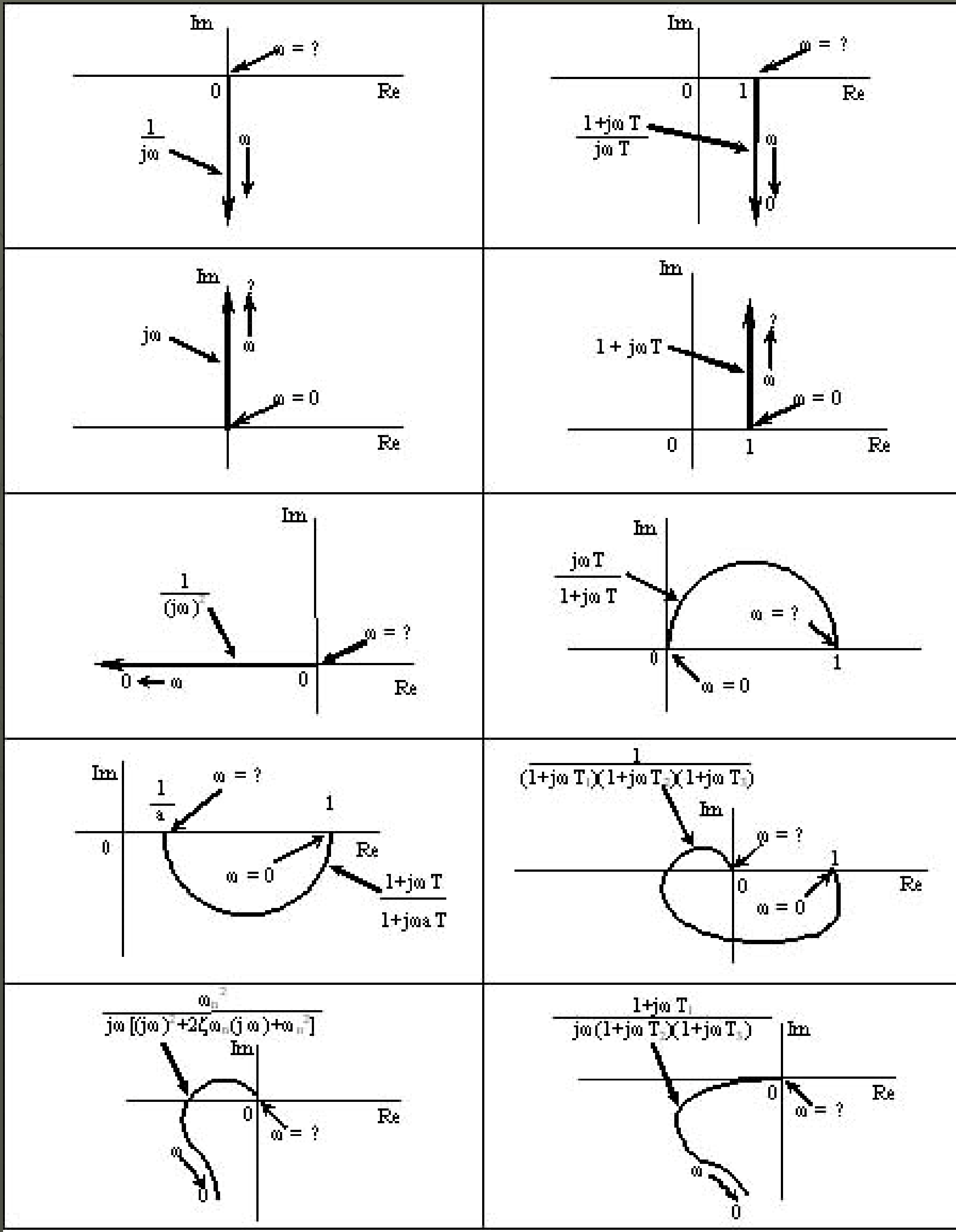
Presenting frequency-response characteristics in graphical forms

- Bode diagram or logarithm plot
- Nyquist plot or polar plot
- Log-magnitude-versus-phase plot (Nichols plots)

Polar Plot



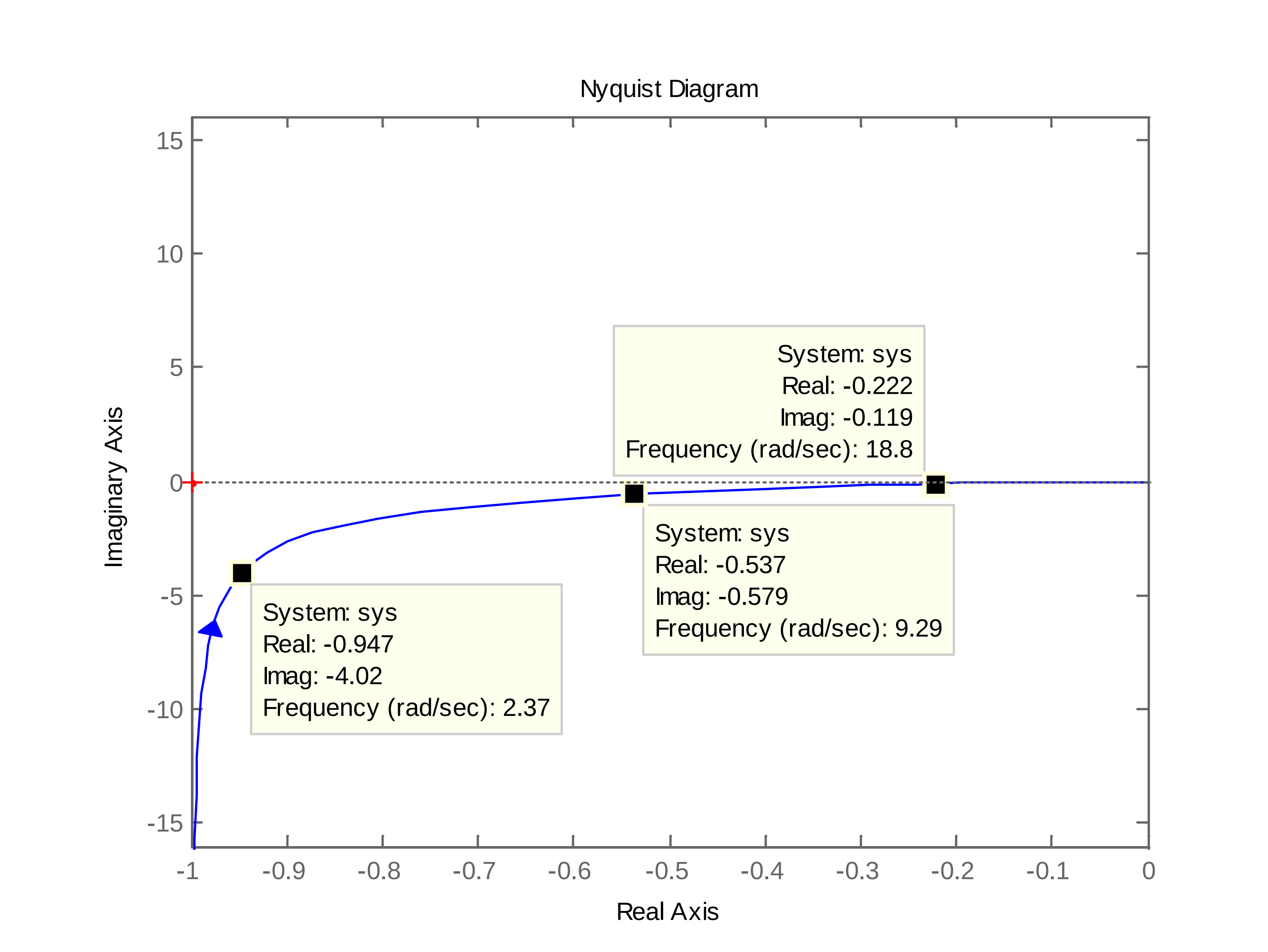
Polar Plot



Im
Re
0
ω = ?
1/jω
ω
1+jωT / jωT
1
jω
ω = 0
1 + jωT
1/(jω)^2
0 ← ω
jωT / 1+jωT
1/a
1+jωT / 1+jωaT
1 / (1+jωT1)(1+jωT2)(1+jωT3)
ωn^2 / jω[(jω)^2+2ζωn(jω)+ωn^2]
1+jωT1 / jω(1+jωT2)(1+jωT3)

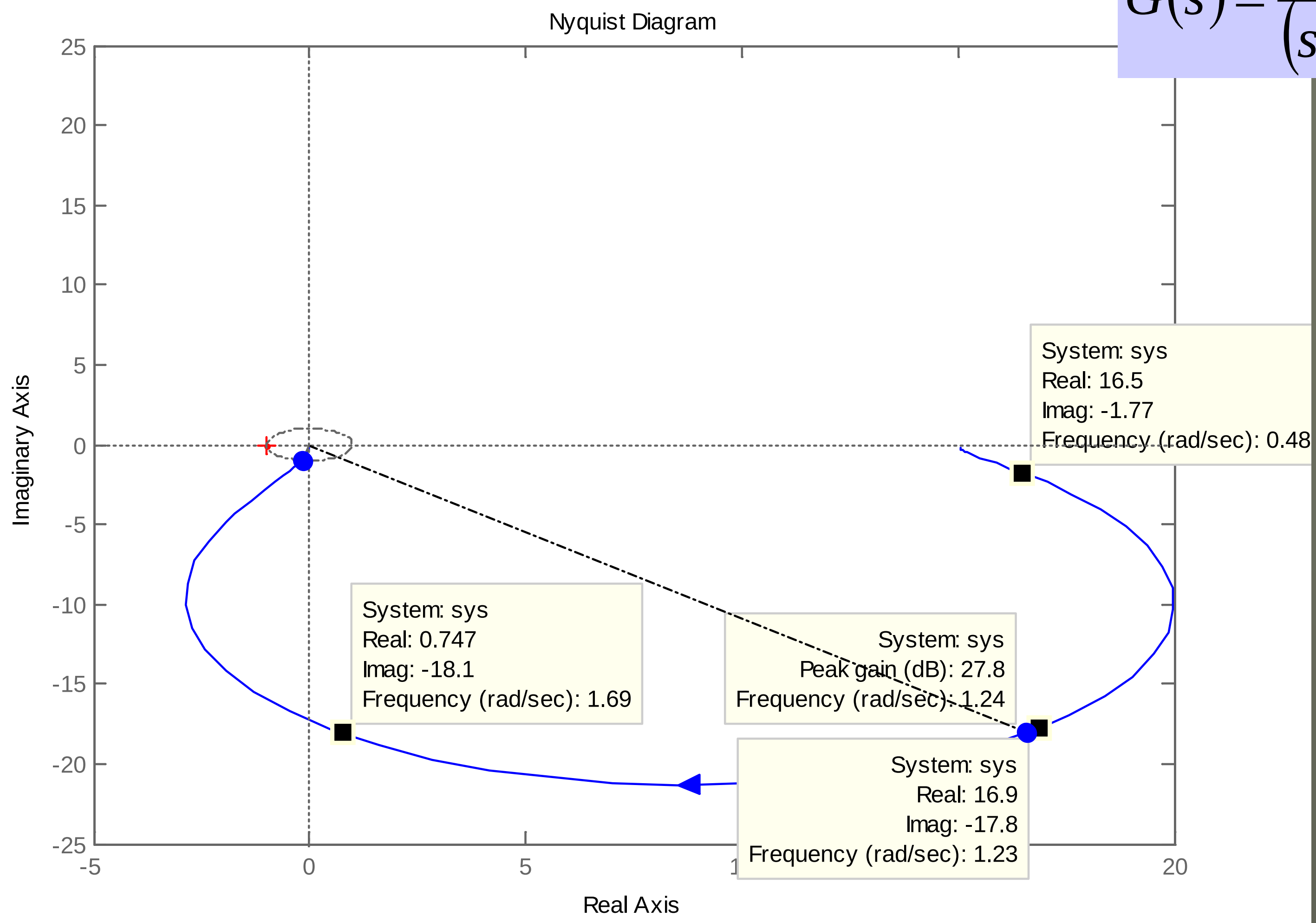
Polar Plot

$$G(s)=\frac{10}{s(0.1s+1)}$$

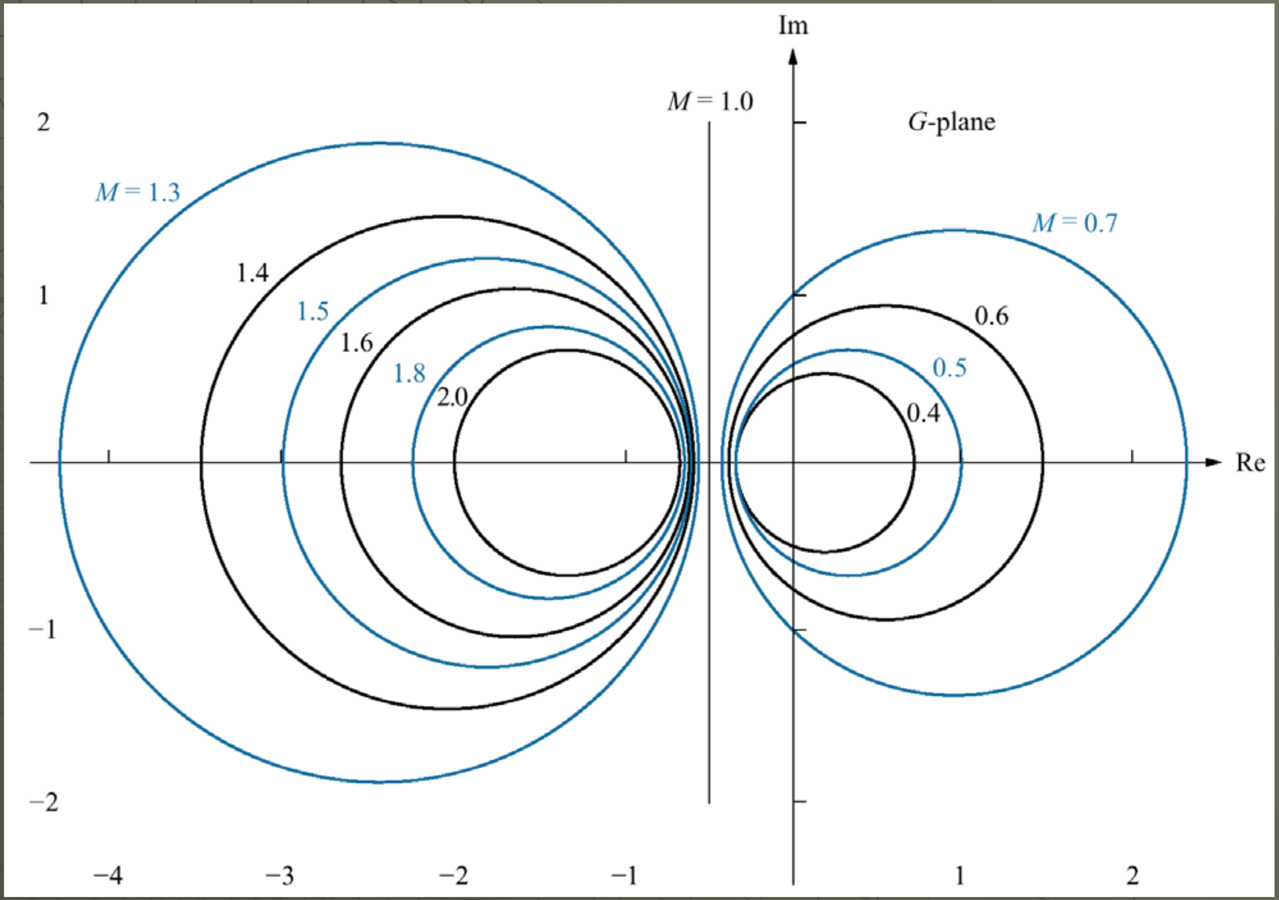


Polar Plot

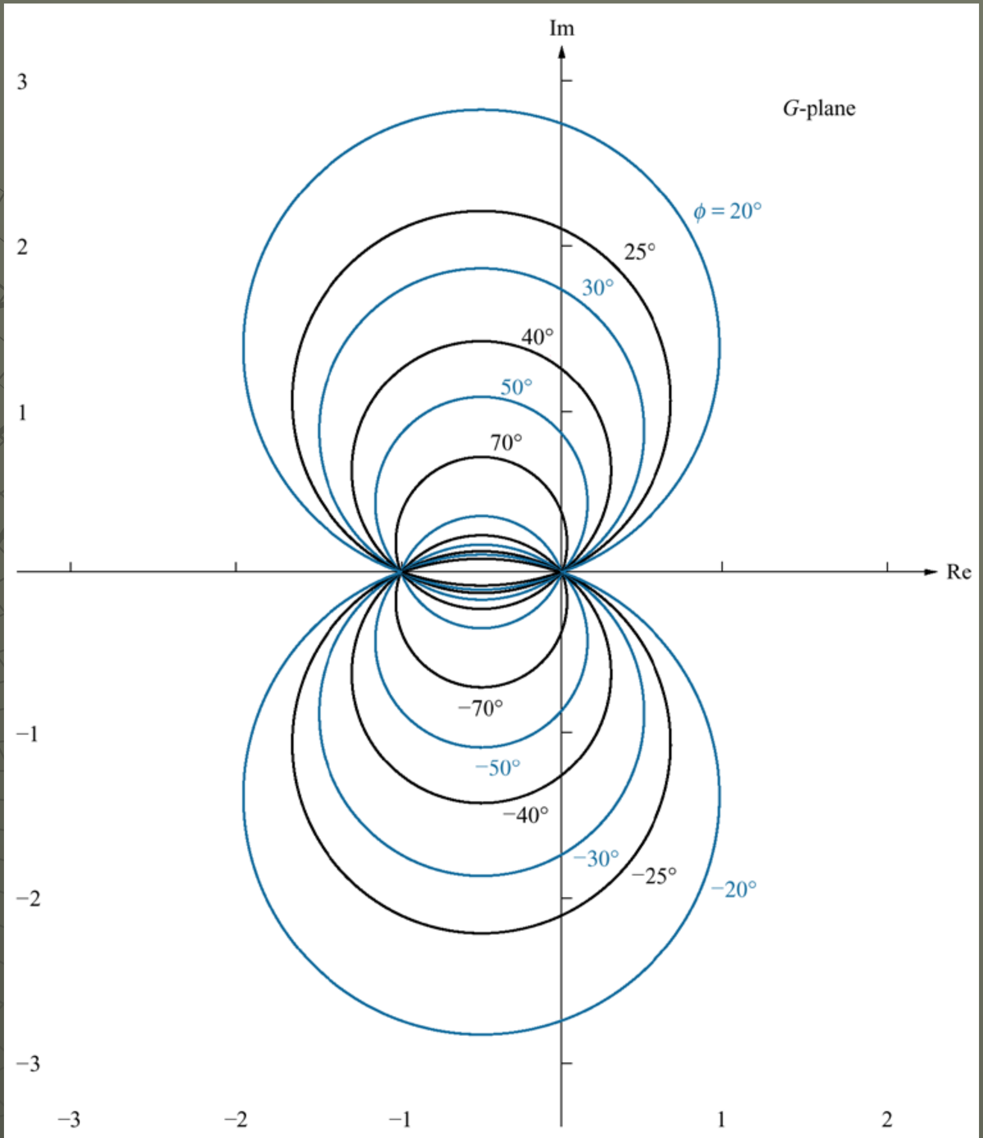
$$G(s)=\frac{10(s+3)}{(s^2+s+2)}$$



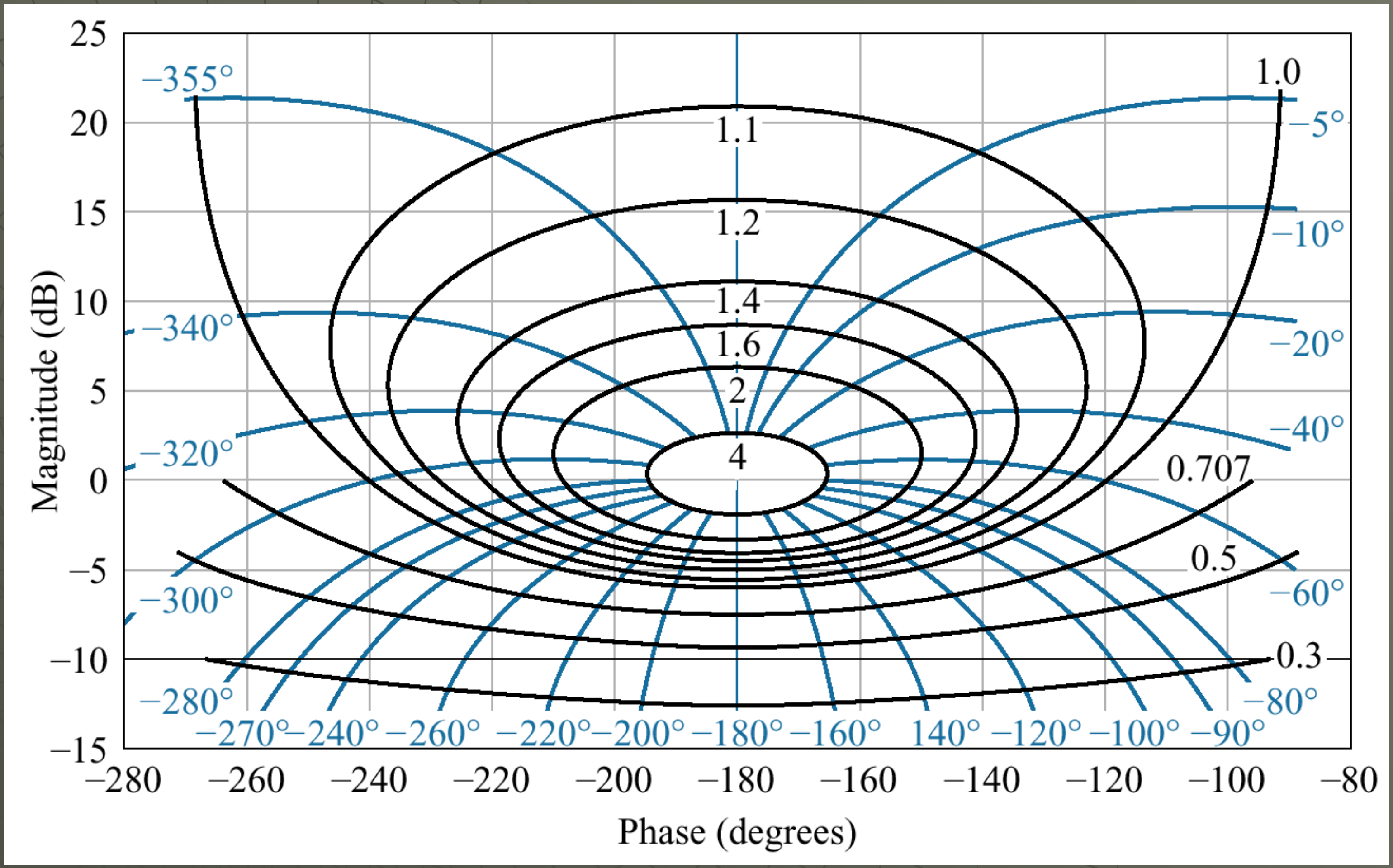
Nichols chart



Nichols chart



Nichols chart



Nichols chart with frequency response for $G(s) = K/[s(s+1)(s+2)]$ superimposed. Values for $K = 1$ and $K = 3.16$